CS 570 Graph Theory

Lecture 5: Coloring

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A **vertex coloring** of a graph $G = (V, E)$ is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called the available **colors**. The smallest integer k such that G has a **k-coloring**, a vertex coloring: $c : V \to \{1, \ldots, k\}$ yields the **chromatic number** of G denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called **k-chromatic**. If $\chi(G) \leq k$, we call G **k-colorable**.

The non-trivial 2-colorable graphs are precisely the bipartite graphs.

An **edge coloring** of $G = (V, E)$ is a map $c : E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e and f. The smallest integer k for which a k-edge coloring exists is the **edge chromatic number** of G denoted by $\chi'(G)$.

Clearly, every edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa ($\chi'(G) = \chi(L(G))$).

1 Coloring maps and planar graphs

Corollary 1.1 ♠ Every planar graph contains a vertex of degree at most five. □

Corollary 1.2 ♠ Every planar graph is 6-colorable. □

Proposition 1.3 [5.1.2][Five Color Theorem] ♠ Every planar graph is 5-colorable. □

Theorem 1.4 [5.1.1][Four Color Theorem] Every planar graph is 4-colorable. □

Theorem 1.5 [5.1.3][Grötzsch 1959] ♠ Every planar graph not containing a triangle is 3-colorable. □

2 Coloring vertices

Proposition 2.1 [5.2.1] ♠ Every graph G with m edges satisfies

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

□

One obvious way to color a graph G in a greedy fashion yields no more than $\Delta(G) + 1$ colors (**how?**). This is rather generous and can be improved:

Proposition 2.2 [5.2.2] Every graph G satisfies

$$\chi(G) \leq 1 + \max\{\delta(H)|\ H \subseteq G\}.$$

$\square$

Corollary 2.3 [5.2.3] Every graph G has a subgraph of minimum degree at least $\chi(G)-1$.
$\square$

Theorem 2.4 [5.2.4][Brooks 1941] Let G be a connected graph. If G is neither complete nor an odd cycle, then

$$\chi(G) \leq \Delta(G).$$

$\square$

3 Coloring edges

Clearly, every graph G satisfies $\chi'(G) \geq \Delta(G)$ (**why?**).

Proposition 3.1 [5.3.1][König 1916] ♠ Every bipartite graph G satisfies

$$\chi'(G) = \Delta(G).$$

$\square$

Theorem 3.2 [5.3.2][Vizing 1964] Every graph G satisfies

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

$\square$