CS 570 Graph Theory

Lecture 2: Matching

Lecturer: Uğur Doğrusöz CS Dept., Bilkent University

A set M of independent edges in a graph $G = (V, E)$ is called a **matching**. M is a matching of $U \subseteq V$ if every vertex in U is incident with an edge in M. The vertices in U are called *matched*. For instance, we have several candidates to fill several openings in a firm. Corresponding to each of n job applicants we can associate a vertex $(a_1, a_2, \ldots, a_n)$, and corresponding to each of m open jobs we can associate a vertex $(j_1, j_2, \ldots, j_m)$. Now we join vertex a_i to vertex j_k if and only if applicant a_i is qualified for job j_k.

A k-regular spanning subgraph is called a **k-factor**. Thus, a subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of V, called **perfect matching**.

1 Matching in bipartite graphs

We let $G = (V, E)$ be a fixed bipartite graph with bipartition $\{A, B\}$, and vertices denoted a, a' will be assumed to lie in A, and vertices denoted b, b' in B.

A path in G which starts in A at an unmatched vertex and then alternately contains edges from $E \backslash M$ and from M, is an **alternating path** with respect to M. An alternating path that ends in an unmatched vertex of B is called an **augmenting path**. Clearly, a graph does not contain any augmenting paths with respect to a maximal matching, since a larger matching can be formed by exchanging the edges in $E \backslash M$ and M (and this method can be used to find optimal matchings).

A set $U \subseteq V$ is called a **cover** of E (or a vertex cover of G) if every edge of G is incident with a vertex in U.

Theorem 1.1 [2.1.1][König] ♠ The maximum cardinality of a matching in G is equal to the minimum cardinality of a vertex cover. □

If G contains a matching of A and $|A| = |B|$, this defines a 1-factor of G, a perfect matching.

A condition clearly necessary for the existence of a matching of A is that every subset of A has enough neighbors in B. The following **marriage theorem** states that this obvious necessary condition is in fact sufficient.

Theorem 1.2 [2.1.2][Hall] ♠ G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$. □

Corollary 1.3 ♠ If $|N(S)| \geq |S| - d$ for every set $S \subseteq A$ and some fixed nonnegative integer d, then G contains a matching of cardinality $|A| - d$. □

Corollary 1.4 [2.1.3] ♠ If G is k-regular with $k \geq 1$, then G has a 1-factor. □

Corollary 1.5 [2.1.5] Every regular graph of positive even degree has a 2-factor. □

2 Matching in general graphs

Given a graph G, let us denote by $\mathcal{C}_G$ the set of its components, and by $q(G)$ the number of its odd components, those of odd order. If G has a 1-factor, then clearly

$$q(G - S) \leq |S|, \text{ for all } S \subseteq V(G),$$

since every odd component of $G - S$ will send a factor edge to S. This necessary condition is also sufficient.

Theorem 2.1 [2.2.1][Tutte] A graph G has a 1-factor iff $q(G-S) \leq |S|$ for all $S \subseteq V(G)$. □

Corollary 2.2 [2.2.2] Every bridgeless cubic graph has a 1-factor. □

A graph $G = (V, E)$ is called **factor-critical** if $G \neq \phi$ and $G - v$ has a 1-factor for every vertex $v \in G$. Then G itself has no one factor because it has odd order. We call a vertex set $S \subseteq V$ **matchable to** $G - S$ if the bipartite graph H_S, which arises from G by contracting the components $C \in \mathcal{C}_{G-S}$ to single vertices and deleting all the edges inside S, contains a matching of S.

Theorem 2.3 [2.2.3] Every graph $G = (V, E)$ contains a vertex set S with the following two properties:

(i) S is matchable to $G - S$;

(ii) every component of $G - S$ is factor-critical.

Given any such set S, the graph G contains a 1-factor iff $|S| = |\mathcal{C}_{G-S}|$. □

3 Path covers

Consider König's theorem (Theorem 1.1 [2.1.1]). If we orient every edge of G from A to B, the theorem tells us how many disjoint directed paths we need in order to cover all vertices of G: every directed path has length 0 or 1 and number of paths of length 1 is maximized.

Here we put the above question more generally: how many paths in a given directed graph will suffice to cover its entire vertex set?

A **path cover** of a directed graph G is a set of disjoint paths in G which together contain all the vertices of G. Let us denote the maximum cardinality of an independent set of vertices in G by $\alpha(G)$.

Theorem 3.1 [2.5.1][Gallai & Milgram] Every directed graph G has a path cover by at most $\alpha(G)$ paths. □

A subset of a partially ordered set $(P, \leq)$ is a **chain** in P if its elements are pairwise comparable; it is an **antichain** if they are pairwise incomparable.

Corollary 3.2 [2.5.2][Dilworth] ♠ In every finite partially ordered set $(P, \leq)$, the minimum number of chains covering P is equal to the maximum cardinality of an antichain in P. □