CS473 - Algorithms I

Lecture 12
Amortized Analysis

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Amortized Analysis

- Consider a sequence of operations, where
  some operations are expensive,
  some others are cheap.

- *Key point*: The time required to perform a sequence of operations is averaged over all operations performed.

- Amortized analysis can be used to show that:
  - The average cost of an operation is small
    even though a single operation might be expensive
    (when we average over a sequence of operations).

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Amortized Analysis vs Average Case Analysis

- Amortized analysis does not use any *probabilistic reasoning*
- Amortized analysis guarantees

  the average performance of each operation in the worst case

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Example: Stack Operations

PUSH (S, x): push object x onto stack

POP(S): pop the top of the stack S and return the popped object

MULTIPOP(S, k):

pop and return the k top objects of the stack S if $|S| \geq k$

or pop and return the entire stack if $|S| < k$

Runtimes:

PUSH(S, x): $\Theta(1)$

POP(S): $\Theta(1)$

MULTIPOP(S, k): $\Theta(\min(|S|, k))$

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Stack Operations: Multipop

MULTIPOP(S, k)

while not StackEmpty(S) **and** $k \neq 0$ **do**

$t \leftarrow$ POP(S)

$k \leftarrow k - 1$

return

Running time:
$\Theta(\min(s, k))$ where $s = |S|$

Runtime Analysis of Stack Operations

- We want to analyze a sequence of n POP, PUSH, and MULTIPOP operations on an initially empty stack.
- What is the worst-case runtime of a MULTIPOP operation?
  $O(n)$ *because the stack size is at most n*
- What is the worst-case runtime of a sequence of n operations?
  $O(n^2)$ *because we may have O(n) MULTIPOPs, each costing O(n)*
- The analysis is correct, but it is not tight!

We can obtain a tighter bound by using amortized analysis.

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Amortized Analysis Techniques

The most common three techniques

- The aggregate method
- The accounting method
- The potential method

If there are several types of operations in a sequence

- The aggregate method assigns
  - The same amortized cost to each operation
- The accounting method and the potential method may assign
  - Different amortized costs to different types of operations

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The Aggregate Method

- Show that sequence of n operations takes
  - Worst case time $T(n)$ in total for all n
- The amortized cost (average cost in the worst case) per operation is therefore $T(n)/n$
- This amortized cost applies to each operation
  - Even when there are several types of operations in the sequence

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The Aggregate Method: Stack Operations

- Aggregate method considers the entire sequence of n operations
  - Although a single MULTIPOP can be expensive
  - Any sequence of n POP, PUSH, and MULTIPOP operations on an initially empty stack can cost at most $O(n)$

Proof: Each object can be popped once for each time it is pushed. Hence the number of times that POP can be called on a nonempty stack including the calls within MULTIPOP is at most the number of PUSH operations, which is at most n

$\Rightarrow$ The amortized cost of an operation is the average $O(n)/n = O(1)$

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Example: Incrementing a Binary Counter

- Implement k-bit binary counter that counts upward from 0
- Store the bits of counter in array A[0..k-1], where

  length(A) = k

  A[0]: the least significant bit

  A[k-1]: the most significant bit

001

- The binary value stored is:

$$x = \sum_{i=0}^{k-1} A[i].2^i$$

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Binary Counter Increment

To add 1 (mod 2^k) to the counter:

INCREMENT(A, k)

$i \leftarrow 0$
while $i < k$ **and** $A[i] = 1$ **do**
    $A[i] \leftarrow 0$
    $i \leftarrow i + 1$
if $i < k$ **then**
    $A[i] \leftarrow 1$
return

Same idea as the hardware implementation of a ripple-carry counter.

e.g. 000010011111 $\Rightarrow$
000010100000

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Binary Counter Increment

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  $i \leftarrow i + 1$
if $i < k$ **then**
  $A[i] \leftarrow 1$
return

Initially, $x = 0$

i.e. $A[i] = 0$ for all $0 \le i \le k$

What is the worst case runtime for INCREMENT(A,k) ?

$\Theta(k)$ when A contains all 1s

What is the worst case runtime of n INCREMENT operations starting from a zero counter?

$O(kn)$

NOT TIGHT!

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The Aggregate Method: Incrementing a Binary Counter

Counter value	[7]	[6]	[5]	[4]	[3]	[2]	[1]	[0]	Incre cost	Total cost
0	0	0	0	0	0	0	0	0		
1	0	0	0	0	0	0	0	1	1	1
2	0	0	0	0	0	0	1	0	2	3
3	0	0	0	0	0	0	1	1	1	4
4	0	0	0	0	0	1	0	0	3	7
5	0	0	0	0	0	1	0	1	1	8
6	0	0	0	0	0	1	1	0	2	10
7	0	0	0	0	0	1	1	1	1	11
8	0	0	0	0	1	0	0	0	4	15
9	0	0	0	0	1	0	0	1	1	16
10	0	0	0	0	1	0	1	0	2	18
11	0	0	0	0	1	0	1	1	1	19

Bits that flip to achieve the next value are shaded

The Aggregate Method: Incrementing a Binary Counter

- Note that, the running time of an increment operation is proportional to the number of bits flipped
- However, all bits are not flipped at each **INCREMENT**

  $A[0]$ flips at each increment operation

  $A[1]$ flips at alternate increment operations

  $A[2]$ flips only once for 4 successive increment operations

  …
- In general, bit $A[i]$ flips $\lfloor n/2^i \rfloor$ times in a sequence of n **INCREMENTs**

The Aggregate Method: Incrementing a Binary Counter

- Therefore, the total number of flips in the sequence is

$$\sum_{i=0}^{k-1} \lfloor n/2^i \rfloor < n \sum_{i=0}^{\infty} 1/2^i = 2n$$

- The amortized cost of each operation is

$$O(n)/n = O(1)$$

The Accounting Method

- We assign different charges to different operations
  - Some operations are charged more than their real cost
  - Some are charged less than their real cost
- The amount charged for an operation is called its amortized cost.
- When the amortized cost of an operation exceeds its actual cost, the difference is assigned to specific objects in the data structure as credit.
- Credit can be used later to help pay for operations of which amortized cost is less than their actual cost.

Example: Accounting Method for Stack Operations

Suppose the unit cost of pushing or popping a stack element is \$1

Let's assign the following amortized costs:

- PUSH: \$2
- POP: \$0
- MULTIPOP: \$0

Notes:

- Amortized cost of MULTIPOP is a constant (0), whereas the actual cost is variable
- All amortized costs are O(1) in this example. In general, amortized costs of different operations may differ asymptotically.

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Example: Accounting Method for Stack Operations

Operation	Amortized Cost	Real Cost	Notes
PUSH(47)	\$2	\$1	\$1 credit stored
PUSH(10)	\$2	\$1	\$1 credit stored
PUSH(39)	\$2	\$1	\$1 credit stored

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Example: Accounting Method for Stack Operations

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PUSH(39)	\$2	\$1	\$1 credit stored
POP()	\$0	\$1	\$1 credit used

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Example: Accounting Method for Stack Operations

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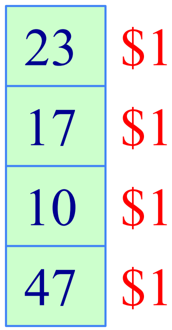
PUSH(39)	$2	$1	$1 credit stored
POP()	$0	$1	$1 credit used
PUSH(17)	$2	$1	$1 credit stored
PUSH(23)	$2	$1	$1 credit stored



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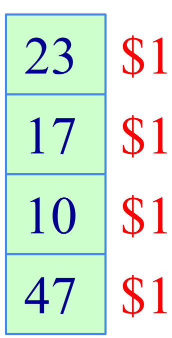
Example: Accounting Method for Stack Operations



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POP()	$0	$1	$1 credit used
PUSH(17)	$2	$1	$1 credit stored
PUSH(23)	$2	$1	$1 credit stored
MULTIPOP(3)	$0	$3	$3 credit used

Example: Accounting Method for Stack Operations

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POP()	\$0	\$1	\$1 credit used
PUSH(17)	\$2	\$1	\$1 credit stored
PUSH(23)	\$2	\$1	\$1 credit stored
MULTIPOP(3)	\$0	\$3	\$3 credit used

47	\$1

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sum of amortized costs ≥ sum of real costs

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Accounting Method for Stack Operations - Notes

- Intuitively:
  - For every PUSH operation, we pay $2:

    $1 for the real cost of PUSH

    $1 pre-payment for the future POP of this item

    (stored as credit)

  - Each POP operation (stand-alone or within MULTI-POP):
    - pays for the real cost by using the credit stored for the corresponding item.

  - The total credit is always nonnegative in a sequence of n operations starting with an empty stack.

The Accounting Method: Stack Operations

Thus by charging the push operation a little bit more we don't need to charge anything from the pop & multipop operations

We have ensured that the amount of credit is always nonnegative

- since each item in the stack always has $1 of credit
- and the stack always has a nonnegative number of items

Thus, for any sequence of *n* push, pop, multipop operations the total amortized cost is an upper bound on the total actual cost

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The Accounting Method (cont'd)

- The amortized cost of an operation can be considered as:

  amortized cost = actual cost + credit

  where credit is either deposited (positive) or used (negative)

- Key point in accounting method:
  - The total amortized cost of a sequence of operations must be an upper bound on the total actual cost.
  - This relationship must hold for all sequences of operations

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} c_i$$

a_i: amortized cost of operation i
c_i: real cost of operation i

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The Accounting Method (cont'd)

For any sequence of n operations, we must have:

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} c_i$$

a_i: amortized cost of operation i

c_i: real cost of operation i

The total credit stored after n operations is:

$$total\_credit = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} c_i$$

For the above inequality to hold, the total credit must be nonnegative at all times.

The Accounting Method - Summary

- Assign an amortized cost for each operation.
- For operation i, let a_i and c_i be the amortized and the real cost of i.

If $a_i > c_i$ $\Rightarrow$ store $(a_i - c_i)$ as credit

If $a_i < c_i$ $\Rightarrow$ use $(c_i - a_i)$ stored credit

- If we never run out of credit in a sequence of n operations, we can say that:

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} c_i$$

a_i: amortized cost of operation i
c_i: real cost of operation i

In other words, the sum of amortized costs for n operations is an upper bound for the sum of real costs.

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Accounting Method Example: Binary Counter Increment

- *Reminder: The running time of an increment operation is proportional to the # of bits flipped.*

- Analyze using accounting method:
  - Charge an amortized cost of $2 to set a bit from 0 to 1, and $0 to set a bit from 1 to 0.
  - Intuition: When a bit is set to 1
    - We use $1 to pay for the actual cost of setting the bit to 1
    - We place the other $1 on the bit as credit.
    - At any point, every 1-bit in the counter has $1 credit on it
    - Hence, we don't need to charge anything to reset a bit to 0
      - We just pay for the reset with the $1 on it.

Accounting Method Example: Binary Counter Increment

Binary Counter	Amortized Cost	Real Cost	Notes
0 0 0 0 0			
	$2	$1	$1 credit stored for bit 0
$1 0 0 0 0 1			
	$0+$2	$1 + $1	$1 credit used for bit 0 $1 credit stored for bit 1
$1 0 0 0 1 0			
	$2	$1	$1 credit stored for bit 0
$1 $1 0 0 0 1 1			
	$0+$0+$2	$1+$1+$1	$1 credit used for bit 0 $1 credit used for bit 1 $1 credit stored for bit 2
$1 0 0 1 0 0			

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Accounting Method Example: Binary Counter Increment

```
INCREMENT(A, k)
    i ← 0
    while i < k and A[i] = 1 do
        A[i] ← 0
        i ← i + 1
    if i < k then
        A[i] ← 1
    return
```

- The amortized cost of setting bits to 0 in the first while loop:

  \$0

  (the real cost is paid by the credits)

- The amortized cost of setting a single bit to 1 at the end:

  \$2

  (\$1 is stored as credit for the bit)

- Total amortized cost for an INCREMENT operation?

  \$2

Accounting Method Example: Binary Counter Increment

- For any sequence of n INCREMENT operations starting from counter value 0:
  - The credit never goes negative

    We have \$1 stored as credit for each bit-1.

    We can use the stored credit to flip each bit to 0.
  - So, we have:

$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} c_i$$

a_i: amortized cost of operation i
c_i: real cost of operation i

In other words, the sum of amortized costs for n operations is an upper bound for the sum of real costs.

Accounting Method Example: Binary Counter Increment

- So, we have showed that:

  For n increment operations:

  the total amortized cost is $O(n)$.

  This amortized cost is an upper bound for the actual cost

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The Potential Method

Accounting method represents prepaid work as credit stored with specific objects in the data structure

Potential method represents the prepaid work as potential energy or just potential that can be released to pay for the future operations

The potential is associated with the data structure as a whole rather than with specific objects within the data structure

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The Potential Method

We start with an initial data structure D_0 and perform n operations.

For $1 \leq i \leq n$, let:

C_i: the actual cost of the i^{th} operation

D_i: data structure that results after applying i^{th} operation to D_{i-1}

ϕ: potential function that maps each data structure D_i to a real number $\phi(D_i)$

$\phi(D_i)$: the potential associated with data structure D_i

$\hat{C}_i$: amortized cost of the i^{th} operation w.r.t. function ϕ

The Potential Method

$$\underbrace{\hat{C}_i = C_i}_{\text{actual cost}} + \underbrace{\phi(D_i) - \phi(D_{i-1})}_{\text{increase in potential due to the operation}}$$

The total amortized cost of n operations is

$$\sum_{i=1}^{n} \hat{C}_i = \sum_{i=1}^{n} (C_i + \phi(D_i) - \phi(D_{i-1}))$$

$$= \sum_{i=1}^{n} C_i + \phi(D_n) - \phi(D_0)$$

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The Potential Method

If we can ensure that $\varphi(D_n) \geq \varphi(D_0)$ then the total amortized cost $\sum_{i=1}^{n} \hat{C}_i$ is an upper bound on the total actual cost

However, $\varphi(D_n) \geq \varphi(D_0)$ should hold for all possible n since, in practice, we do not always know n in advance

Hence, if we require that $\varphi(D_i) \geq \varphi(D_0)$, for all i, then we ensure that the total amortized cost is an upper bound for the total cost

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The Potential Method: Stack Operations

- Define $\varphi(S) = |S|$, the number of objects in the stack
- For the initially empty stack, we have $\varphi(D_0) = 0$
- Since $|S| \geq 0$, stack D_i that results after ith operation has nonnegative potential for all i, that is

$$\varphi(D_i) \geq 0 = \varphi(D_0) \text{ for all } i$$

- Hence, the total amortized cost is an upper bound on the total actual cost

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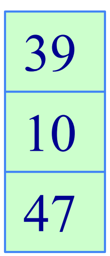
The Potential Method for Stack Operations

Operation	Real Cost	Potential	Amortized Cost
		0	
PUSH(47)	1	1	2
PUSH(10)	1	2	2
PUSH(39)	1	3	2



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The Potential Method for Stack Operations

Operation	Real Cost	Potential	Amortized Cost
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PUSH(10)	1	2	2
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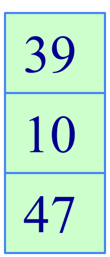
POP()	1	2	0



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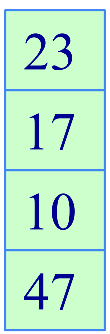
The Potential Method for Stack Operations

Operation	Real Cost	Potential	Amortized Cost
		0	
PUSH(47)	1	1	2
PUSH(10)	1	2	2
PUSH(39)	1	3	2
POP()	1	2	0
PUSH(17)	1	3	2
PUSH(23)	1	4	2



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The Potential Method for Stack Operations

Operation	Real Cost	Potential	Amortized Cost
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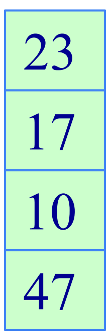
PUSH(10)	1	2	2
PUSH(39)	1	3	2
POP()	1	2	0
PUSH(17)	1	3	2
PUSH(23)	1	4	2
MULTIPOP(3)	3	1	0



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The Potential Method for Stack Operations

Operation	Real Cost	Potential	Amortized Cost
		0	
PUSH(47)	1	1	2
PUSH(10)	1	2	2
PUSH(39)	1	3	2
POP()	1	2	0
PUSH(17)	1	3	2
PUSH(23)	1	4	2
MULTIPOP(3)	3	1	0

47

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sum of amortized costs $\geq$ sum of real costs

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The Potential Method for Stack Operations

Reminder: $\phi(D_i)$: The number of objects in stack after operation i

PUSH(S, x):

$\phi(D_i) - \phi(D_{i-1}) = 1$ *(because the stack size increases by 1)*

$\hat{C}_i = C_i + \phi(D_i) - \phi(D_{i-1}) = 1 + 1 = 2$

Amortized cost of PUSH operation is 2

POP(S):

$\phi(D_i) - \phi(D_{i-1}) = -1$ *(because the stack size decreases by 1)*

$\hat{C}_i = C_i + \phi(D_i) - \phi(D_{i-1}) = 1 - 1 = 0$

Amortized cost of POP operation is 0

The Potential Method for Stack Operations

Reminder: $\phi(D_i)$: The number of objects in stack after operation i

MULTIPOP(S, k):

$\phi(D_i) - \phi(D_{i-1}) = -k'$, where $k' = \min\{|S|, k\}$

because the stack size decreases by k′

$\hat{C}_i = C_i + \phi(D_i) - \phi(D_{i-1}) = k' - k' = 0$

Amortized cost of MULTIPOP operation is 0

The amortized cost of each operation is O(1).

Thus, the amortized cost of a sequence of n operations is O(n)

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The Potential Method - Intuition

If $\phi(D_i) - \phi(D_{i-1}) > 0$, then:

Amortized cost $\hat{C}_i$ is an overcharge for the i^{th} operation.

The potential of the data structure increases.

If $\phi(D_i) - \phi(D_{i-1}) < 0$, then:

Amortized cost $\hat{C}_i$ is an undercharge for the i^{th} operation.

The actual cost of the operation is paid by the decrease in potential.

The Potential Method - Intuition

Different potential functions may yield different amortized costs.

The best potential function to use depends on the desired time bounds.

Choose a potential function such that $\phi(D_i) - \phi(D_0) \geq 0$ for all i values. This ensures that the amortized cost of any i operations is an upper bound for the actual cost.

Practical guideline:

Choose a potential function that increases a little after every cheap operation, and decreases a lot after an expensive operation.

The Potential Method for Binary Counter Increment

- Define $\phi(D_i) = b_i$

  where b_i: number of 1s in the counter after the i^{th} operation

- The actual cost of INCREMENT operation:

  C_i = (# bits changed 0⟹1) + (# of bits changed 1⟹0)

- The potential change after the i^{th} INCREMENT operation:

  $\phi(D_i) - \phi(D_{i-1})$ = (# of bits changed 0⟹1) − (# of bits changed 1⟹0)

- Amortized cost of the i^{th} INCREMENT operation:

  $\hat{C}_i = C_i + \phi(D_i) - \phi(D_{i-1})$

  $= 2$. (# of bits changed from 0 ⟹ 1)

The Potential Method for Binary Counter Increment

- Amortized cost of the i^{th} INCREMENT operation:

  $$\hat{C}_i = 2 \ . \ (\#\text{ of bits changed from } 0 \Rightarrow 1)$$

- In one INCREMENT operation, we change at most 1 bit $0 \Rightarrow 1$

- Hence, the amortized cost of an INCREMENT operation:

  $$\hat{C}_i \leq 2$$

The Potential Method for Binary Counter Increment

Binary Counter	Real Cost	Potential	Amortized Cost
0 0 0 0 0		0	
0 0 0 0 1	1	1	2
0 0 0 1 0	2	1	2
0 0 0 1 1	1	2	2
0 0 1 0 0	3	1	2

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The Potential Method: Incrementing a Binary Counter

- If the counter starts at zero, then $\varphi(D_0) = 0$, the number of 1s in the counter after the ith operation
- Since $\varphi(D_i) \geq 0$ for all i the total amortized cost is an upper bound on the total actual cost
- Hence, the worst-case cost of n operations is $O(n)$

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The Potential Method for Binary Counter Increment

- What if the counter does not start from zero *(i.e. $b_0 \neq 0$)*?

- For a sequence of n INCREMENT operations, can we say that the sum of the amortized costs is an upper bound for the sum of the actual costs?

  No, because:
$$\sum_{i=1}^{n} \hat{C}_i = \sum_{i=1}^{n} C_i + \phi(D_n) - \phi(D_0)$$
  and $\phi(D_0) = b_0 \neq 0$.

  So, $\phi(D_n) - \phi(D_0)$ is not necessarily ≥ 0

Reminder: $\phi(D_i) = b_i$

where b_i: number of 1s in the counter after the i^{th} operation

The Potential Method for Binary Counter Increment

- What if the counter does not start from zero *(i.e.* $b_0 \neq 0$*)*?

- For a sequence of n INCREMENT operations we can write:

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} \hat{C}_i - \phi(D_n) + \phi(D_0)$$

$$\leq 2n - b_n + b_0 \qquad \textit{(because } \hat{C}_i \leq 2 \textit{ for all i)}$$

Note: $b_0 \leq k$, where k is the number of bits of the counter.

If we execute at least $n = \Omega(k)$ INCREMENT operations, the total actual cost will be O(n), no matter what the initial counter value is.

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Amortized Analysis - Summary

- With amortized analysis, we show that the average cost of an operation is small if we average over a sequence of operations (even though some single operations may be expensive).

- We studied 3 techniques for amortized analysis:
  - Aggregate Method
  - Accounting Method
  - Potential Method

Amortized Analysis - Summary

- ***Aggregate Method***:
  - Directly compute the sum of n operations.
  - Then, compute the average.

- ***Accounting Method:***
  - Pay a little extra for the cheap operations and
  store the difference as credit on certain items
  - Pay for the expensive operations using the stored credit.
  - As long as we never run of out of credits:
    - The total amortized cost is guaranteed to be an upper bound for the total actual cost.

Amortized Analysis - Summary

- ***Potential Method:***
  - Similar to the accounting method, but a potential function is defined for the whole data structure instead of individual items.
  - The potential is 0 initially.
  - It increases slowly with every cheap operation.
  - Expensive operations are paid using the potential stored.
  - Amortized cost is the actual cost plus the change in potential.
  - As long as potential is always nonnegative:

The total amortized cost is guaranteed to be an upper bound for the total actual cost.