CS473 - Algorithms I

Lecture 7

Medians and Order Statistics

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Medians and Order Statistics

i^{th} order statistic: i^{th} smallest element of a set of n elements

minimum: first order statistic

maximum: n^{th} order statistic

median: "halfway point" of the set

$$i = \lfloor (n+1)/2 \rfloor \text{ or } \lceil (n+1)/2 \rceil$$

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Selection Problem

- *Selection problem*: Select the i^{th} smallest of n elements

- *Naïve algorithm*: Sort the input array A; then return A[i]

  $T(n) = \Theta(n\lg n)$

  using e.g. merge sort (but not quicksort)

- Can we do any better?

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Selection in Expected Linear Time

- Randomized algorithm using divide and conquer
- Similar to randomized quicksort
  - *Like quicksort*: Partitions input array recursively
  - *Unlike quicksort*: Makes a single recursive call

    Reminder: *Quicksort makes two recursive calls*
- Expected runtime: $\Theta(n)$

  Reminder: *Expected runtime of quicksort:* $\Theta(n\lg n)$

Selection in Expected Linear Time: Example 1

Select the 2nd smallest element:

6	10	13	5	8	3	2	11

$i = 2$

Partition the input array:

2	3	5	13	8	10	6	11

make a recursive call to select the 2nd smallest element in left subarray

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Selection in Expected Linear Time: Example 2

Select the 7^{th} smallest element:

6	10	13	5	8	3	2	11

$i = 7$

Partition the input array:

2	3	5	13	8	10	6	11

make a recursive call to select the 4^{th} smallest element in right subarray

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Selection in Expected Linear Time

```
R-SELECT(A,p,r,i)
    if p = r then
        return A[p]
  q ← R-PARTITION(A, p, r)
  k ← q–p+1
  if i ≤ k then
        return R-SELECT(A, p, q, i)
else
        return R-SELECT(A, q+1, r, i-k)
```

$\leq x$ (k smallest elements)	$\geq x$

p $\qquad$ q $\qquad$ r

$x = pivot$

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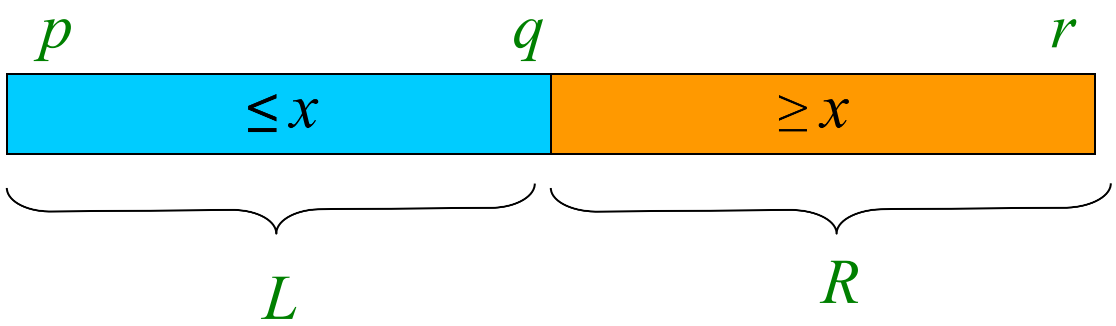
Selection in Expected Linear Time



$x = pivot$

- All elements in L ≤ all elements in R
- L contains $|L| = q-p+1 = k$ smallest elements of $A[p \ldots r]$

  if $i \leq |L| = k$ then

  search L recursively for its i-th smallest element

  else

  search R recursively for its $(i-k)$-th smallest element

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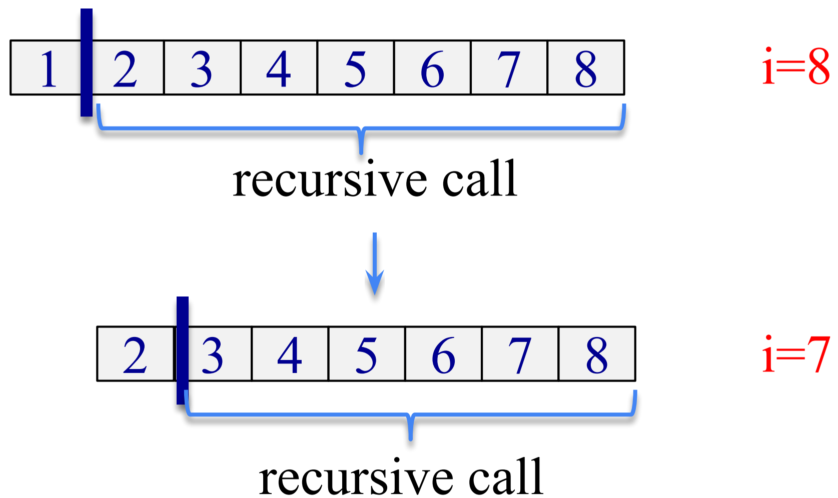
Runtime Analysis

- ***Worst case:***

  Imbalanced partitioning at every level

  and the recursive call always to the larger partition



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Runtime Analysis

- ***Worst case:***

  $T(n) = T(n\text{-}1) + \Theta(n)$

  $\Rightarrow T(n) = \Theta(n^2)$

  Worse than the naïve method (based on sorting)

- ***Best case***: Balanced partitioning at every recursive level

  $T(n) = T(n/2) + \Theta(n)$

  $\Rightarrow T(n) = \Theta(n)$

- ***Avg case***: Expected runtime – need analysis

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Reminder: Various Outcomes of H-PARTITION

$P(\text{rank}(x) = i) = 1/n$ for $1 \le i \le n$

x: pivot

|L|: size of left region

if $\text{rank}(x) = 1$ **then** $|L| = 1$

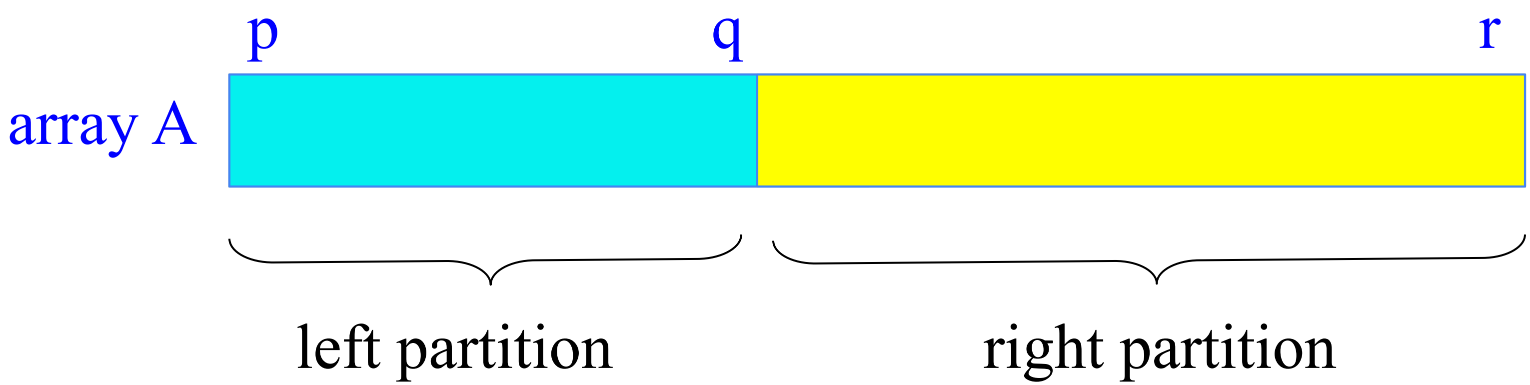
if $\text{rank}(x) > 1$ **then** $|L| = \text{rank}(x) - 1$

$\boxed{P(|L| = 1) = P(\text{rank}(x) = 1) + P(\text{rank}(x) = 2)}$ $\Rightarrow$ $P(|L| = 1) = 2/n$

$\boxed{P(|L| = i) = P(\text{rank}(x) = i+1) \quad \text{for } 1 < i < n}$ $\Rightarrow$ $P(|L| = i) = 1/n$ for $1 < i < n$

Average Case Analysis of Randomized Select

- To compute the upper bound for the avg case, assume that the i^{th} element always falls into the larger partition.

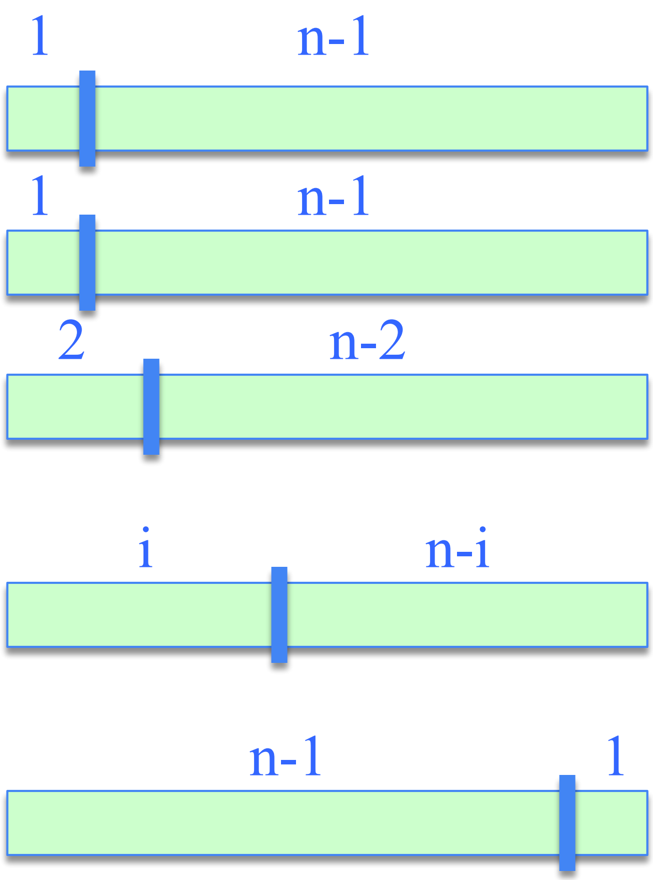


We will analyze the case where the recursive call is always made to the larger partition

□ this will give us an upper bound for the avg case

Various Outcomes of H-PARTITION

rank(x)	prob.	T(n)
1	1/n	$\leq T(\max(1, n-1)) + \Theta(n)$
2	1/n	$\leq T(\max(1, n-1)) + \Theta(n)$
3	1/n	$\leq T(\max(2, n-2)) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
i+1	1/n	$\leq T(\max(i, n-i)) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
n	1/n	$\leq T(\max(n-1, 1)) + \Theta(n)$



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Average-Case Analysis of Randomized Select

Recall: $P(|L|=i) = \begin{cases} 2/n & \text{for } i = 1 \\ 1/n & \text{for } i = 2,3,\ldots,n-1 \end{cases}$

Upper bound: Assume i-th element always falls into the larger part

$$T(n) \leq \frac{1}{n} T(\max(1, n-1)) + \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

Note: $\frac{1}{n} T(\max(1, n-1)) = \frac{1}{n} T(n-1) = \frac{1}{n} O(n^2) = O(n)$

$$\therefore \quad T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

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Average-Case Analysis of Randomized Select

$$\therefore\ T(n) \le \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\max(q, n-q) = \begin{cases} q & \text{if } q \ge \lceil n/2 \rceil \\ n-q & \text{if } q < \lceil n/2 \rceil \end{cases}$$

n is odd: $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n-1$

n is even: $T(\lceil n/2 \rceil)$ appears once $T(k)$ appears twice for

$k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n-1$

Hence, in both cases: $\sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n) \le 2 \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$

$$\therefore\ T(n) \le \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

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Average-Case Analysis of Randomized Select

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

By substitution guess $T(n) = O(n)$

Inductive hypothesis: $T(k) \leq ck, \ \forall\, k < n$

$$T(n) \leq (2/n) \sum_{k=\lceil n/2 \rceil}^{n-1} ck + O(n)$$

$$= \frac{2c}{n} \left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \right) + O(n)$$

$$\frac{2c}{n} \left(\frac{1}{2} n\,(n-1) - \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \left(\frac{n}{2} - 1 \right) \right) + O(n)$$

Average-Case Analysis of Randomized Select

$$T(n) \le \frac{2c}{n}\left(\frac{1}{2}n(n\text{-}1) - \frac{1}{2}\left\lceil\frac{n}{2}\right\rceil\left(\frac{n}{2} - 1\right)\right) + \mathrm{O}(n)$$

$$\le c(n\text{-}1) - \frac{c}{4}n + \frac{c}{2} + \mathrm{O}(n)$$

$$= cn - \frac{c}{4}n - \frac{c}{2} + \mathrm{O}(n)$$

$$= cn - \left(\left(\frac{c}{4}n + \frac{c}{2}\right) - \mathrm{O}(n)\right)$$

$$\le cn$$

since we can choose c large enough so that **($cn/4+c/2$) dominates O(n)**

Summary of Randomized Order-Statistic Selection

- Works fast: linear expected time
- Excellent algorithm in practise
- But, the worst case is very bad: $\Theta(n^2)$

Q: Is there an algorithm that runs in linear time in the worst case?

A: Yes, due to Blum, Floyd, Pratt, Rivest & Tarjan [1973]

Idea: Generate a good pivot recursively..

Selection in Worst Case Linear Time

SELECT(S, n, i) return $\boldsymbol{i}$-th element in set S with $\boldsymbol{n}$ elements

if $n \le 5$ then

SORT S and return the i-th element

DIVIDE S into $\lceil n/5 \rceil$ groups

first $\lceil n/5 \rceil$ groups are of size 5, last group is of size n mod 5

FIND median set $M=\{m_1, \ldots, m_{\lceil n/5 \rceil}\}$ m_j : median of j-th group

$x \leftarrow$ SELECT(M, $\lceil n/5 \rceil$, $(\lfloor \lceil n/5 \rceil + 1)/2 \rfloor$)

PARTITION set S around the pivot x into L and R

if $i \le |L|$ then

return SELECT(L, $|L|$, i)

else

return SELECT(R, $n-|L|$, $i-|L|$)

Selection in Worst Case Linear Time - Example

Input: Array S and index i

Output: The i^{th} smallest value

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41
2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

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Selection in Worst Case Linear Time - Example

Step 1: Divide the input array into groups of size 5

25	27	32	22	30	7	37	18	26
9	39	14	31	41	21	23	24	35
16	42	36	4	2	10	40	12	43
8	15	20	17	13	34	5	38	
11	6	33	3	19	1	29	28	

Selection in Worst Case Linear Time - Example

Step 2: Compute the median of each group $\Rightarrow \Theta(n)$

9	15	14	4	2	7	5	18	
8	6	20	3	13	1	23	12	26
11	**27**	**32**	**17**	**19**	**10**	**29**	**24**	**35**
16	42	33	31	30	34	40	28	43
25	39	36	22	41	21	37	38	

Let M be the set of the medians computed:

$M = \{11, 27, 32, 17, 19, 10, 29, 24, 35\}$

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Selection in Worst Case Linear Time - Example

Step 3: Compute the median of the median group M

$x \leftarrow$ SELECT (M, |M|, $\lfloor (|M|+1)/2 \rfloor$) where $|M| = \lceil n/5 \rceil$

	9	15	14	4	2	7	5	18	
	8	6	20	3	13	1	23	12	26
M	11	27	32	17	19	10	29	24	35
	16	42	33	31	30	34	40	28	43
	25	39	36	22	41	21	37	38	

$\Rightarrow$ median = 24

The runtime of the recursive call: T(|M|) $= T(\lceil n/5 \rceil)$

Selection in Worst Case Linear Time - Example

Step 4: Partition the input array S around the median-of-medians x

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41
2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

Partition S around x = 24

Claim: Partitioning around x is guaranteed to be *well-balanced*.

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Selection in Worst Case Linear Time - Example

Claim: Partitioning around x=24 is guaranteed to be *well-balanced*.

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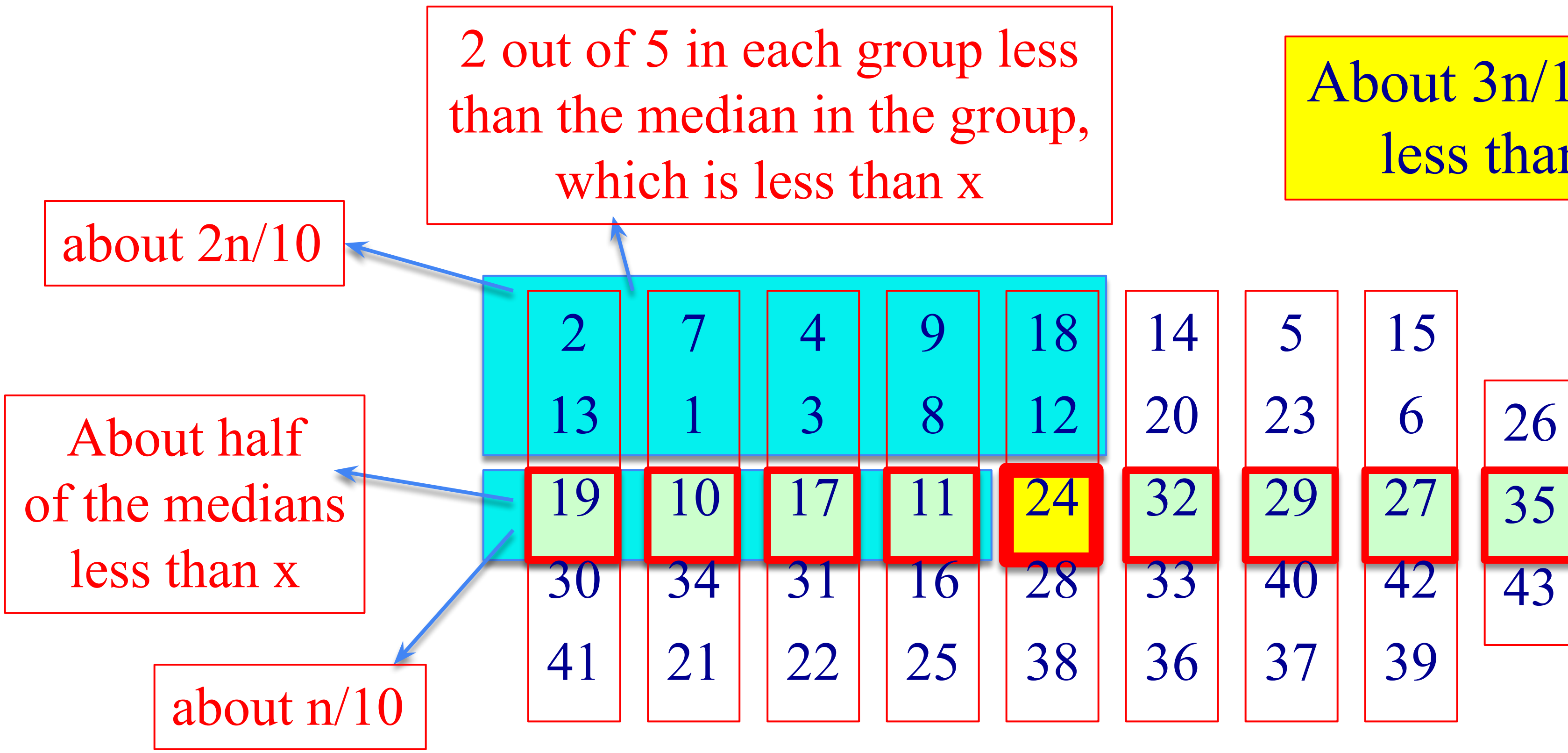
Selection in Worst Case Linear Time - Example

Claim: Partitioning around x=24 is guaranteed to be *well-balanced*.



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Selection in Worst Case Linear Time - Example

S = {25 9 16 8 11 27 39 42 15 6 32 14 36 20 33 22 31 4 17 3 30 41 2 13 19 7 21 10 34 1 37 23 40 5 29 18 24 12 38 28 26 35 43}

Partitioning S around x = 24 will lead to partitions of sizes ~3n/10 and ~7n/10 in the worst case.

Step 5: Make a recursive call to one of the partitions

if $i \le |L|$ then
    return SELECT(L, $|L|$, i)
else
    return SELECT(R, n–$|L|$, i–$|L|$)

Selection in Worst Case Linear Time

```
SELECT(S, n, i)      return i-th element in set S with n elements
    if n ≤ 5 then
        SORT S and return the i-th element
    DIVIDE S into ⌈n/5⌉groups
      first ⌈n/5⌉ groups are of size 5, last group is of size n mod 5
    FIND median set M={m_1 , …, m_⌈n/5⌉ }    m_j : median of j-th group
    x ← SELECT(M, ⌈n/5⌉, (⌊⌈n/5⌉+1)/2⌋)
    PARTITION set S around the pivot x into L and R
    if i ≤ |L| then
        return  SELECT(L, |L|, i)
    else
        return  SELECT(R, n–|L|, i–|L|)
```


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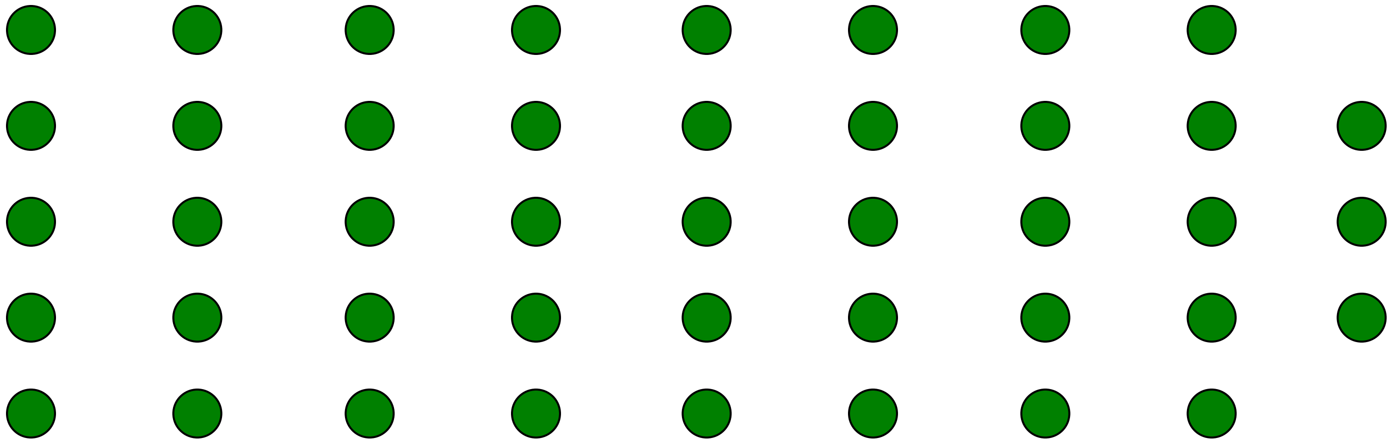
Choosing the Pivot



1. Divide S into groups of size 5

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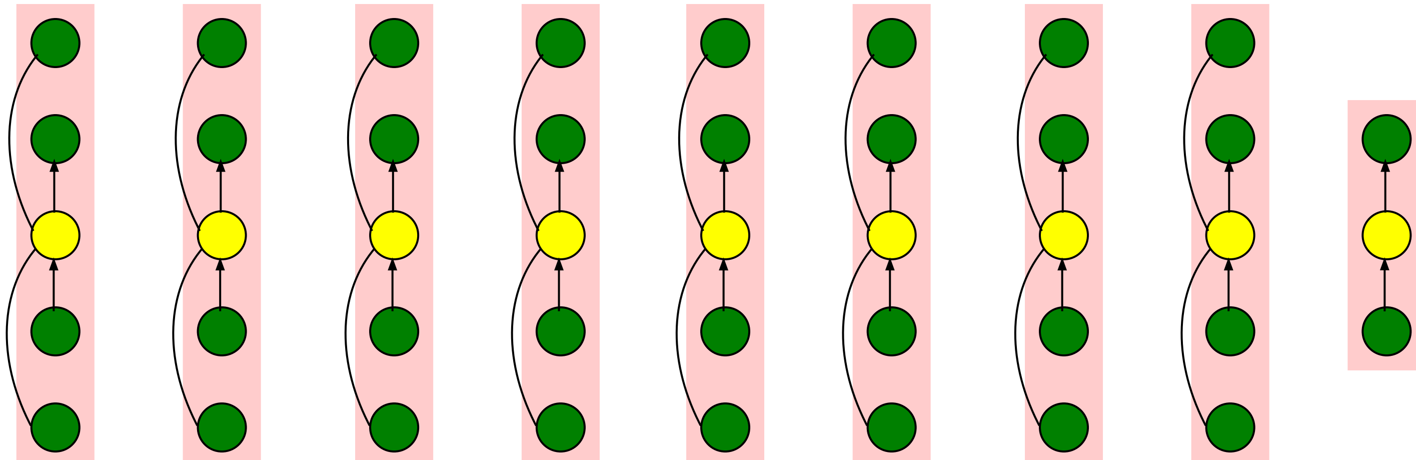
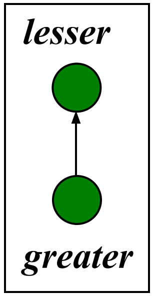
Choosing the Pivot



1. Divide S into groups of size 5
2. Find the median of each group



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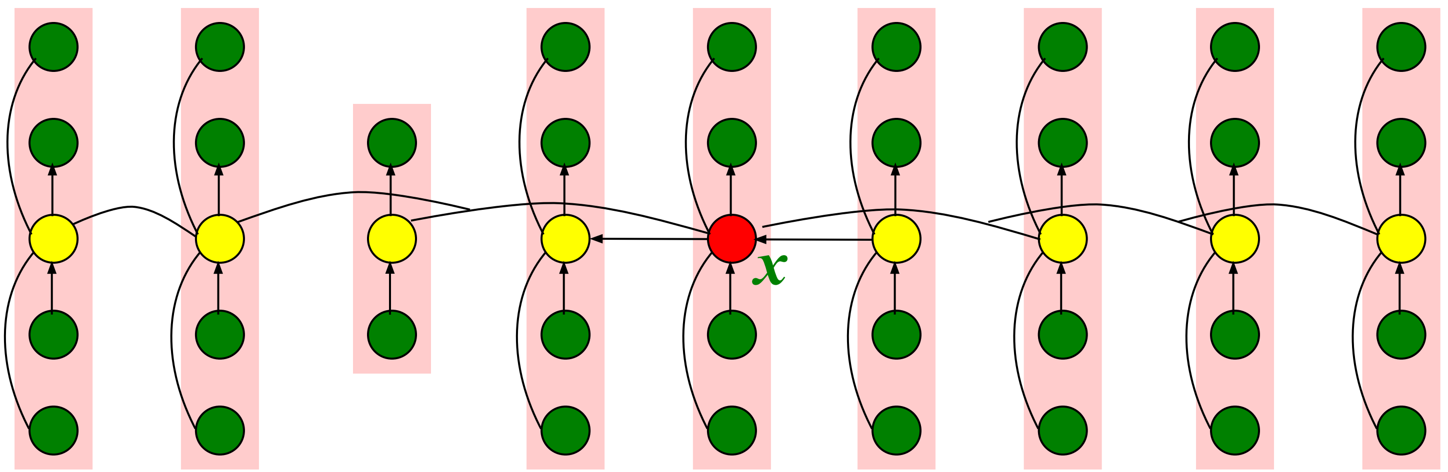
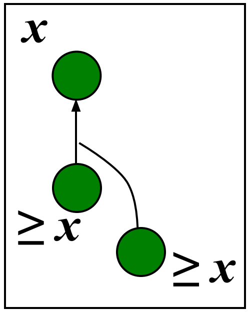
Choosing the Pivot



1. Divide S into groups of size 5
2. Find the median of each group
3. Recursively select the median x of the medians



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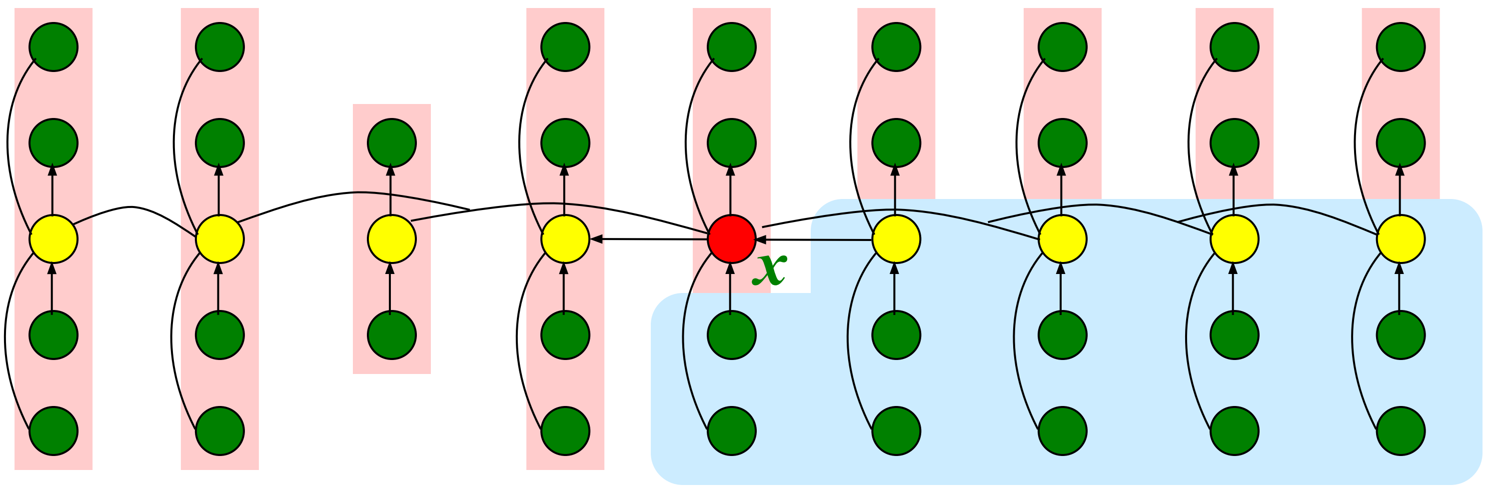
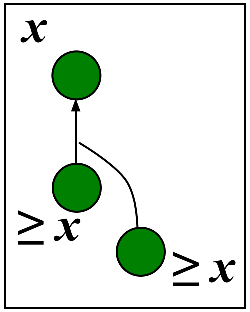
Choosing the Pivot



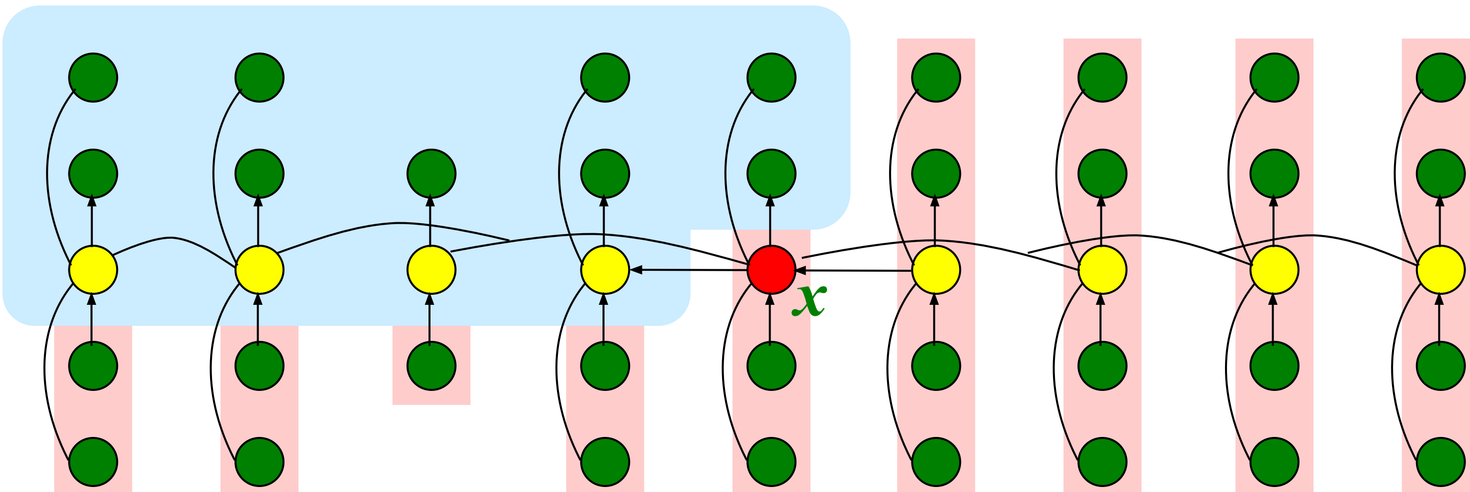
At least half of the medians $\geq x$

Thus $m = \left\lceil \lceil n/5 \rceil / 2 \right\rceil$ groups contribute 3 elements to R except possibly the last group and the group that contains x

$$|R| \geq 3\left(m - 2\right) \geq \frac{3n}{10} - 6$$



Analysis

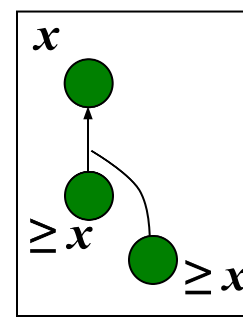


Similarly

$|L| \geq \frac{3n}{10} - 6$

Therefore, SELECT is recursively called on at most

$n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$ elements



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Selection in Worst Case Linear Time

SELECT(S, n, i)   return ***i***-th element in set S with ***n*** elements

$\Theta(n)$ {
if $n \le 5$ then
  SORT S and return the i-th element
DIVIDE S into $\lceil n/5 \rceil$ groups
  first $\lceil n/5 \rceil$ groups are of size 5, last group is of size n mod 5
}

$\Theta(n)$ { FIND median set $M = \{m_1, \ldots, m_{\lceil n/5 \rceil}\}$ m_j : median of j-th group

$T(\lceil n/5 \rceil)$ { $x \leftarrow$ SELECT(M, $\lceil n/5 \rceil$, $(\lfloor \lceil n/5 \rceil + 1)/2 \rfloor$)

$\Theta(n)$ { PARTITION set S around the pivot x into L and R

$T(\frac{7n}{10} + 6)$ {
if $i \le |L|$ then
  return SELECT(L, $|L|$, i)
else
  return SELECT(R, $n - |L|$, $i - |L|$)
}

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Selection in Worst Case Linear Time

Thus recurrence becomes

$$T(n) \le T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n)$$

Guess $T(n) = O(n)$ and prove by induction

Inductive step: $T(n) \le c\lceil n/5 \rceil + c\,(7n/10+6) + \Theta(n)$

$\le cn/5 + c + 7cn/10 + 6c + \Theta(n)$

$= 9cn/10 + 7c + \Theta(n)$

$= cn - [c(n/10 - 7) - \Theta(n)] \le cn$ for large c

Work at each level of recursion is a constant factor (9/10) smaller