CS473 - Algorithms I

Lecture 6-a

Analysis of Quicksort

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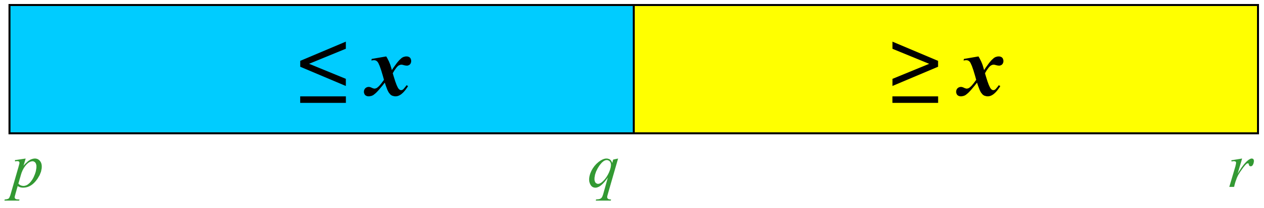
Analysis of Quicksort

Assume ***all elements are distinct*** in the following analysis

```
QUICKSORT (A, $p$, $r$)
        if $p < r$ then
            $q \leftarrow$ H-PARTITION(A, $p$, $r$)
            QUICKSORT(A, $p$, $q$)
            QUICKSORT(A, $q$ +1, $r$)
```



Question

```
QUICKSORT (A, $p$, $r$)
        if $p < r$ then
            $q \leftarrow$ H-PARTITION(A, $p$, $r$)
            QUICKSORT(A, $p$, $q$)
            QUICKSORT(A, $q$ +1, $r$)
```

Q: Remember that H-PARTITION always chooses A[p] *(the first element)* as the **pivot**. What is the runtime of QUICKSORT on an already-sorted array?

✖ a) $\Theta(n)$

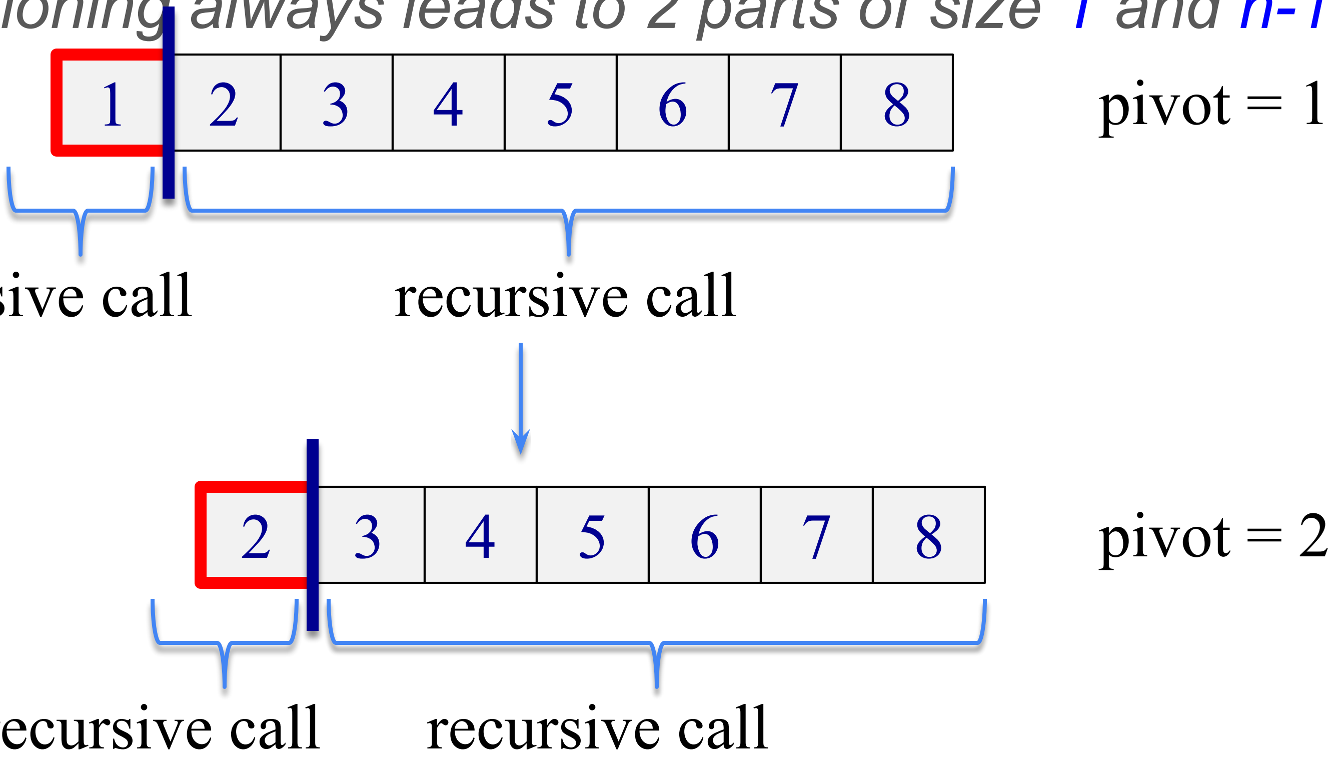
✖ b) $\Theta(n\log n)$

✔ c) $\Theta(n^2)$

✖ d) cannot provide a tight bound

Example: An Already Sorted Array

Partitioning always leads to 2 parts of size 1 and n-1



Worst Case Analysis of Quicksort

- Worst case is when the PARTITION algorithm always returns imbalanced partitions *(of size 1 and n-1)* in every recursive call
  - This happens when the pivot is selected to be either the min or max element.
  - This happens for H-PARTITION when the input array is already sorted or reverse sorted

$$\begin{aligned} T(n) &= T(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textit{(arithmetic series)} \end{aligned}$$

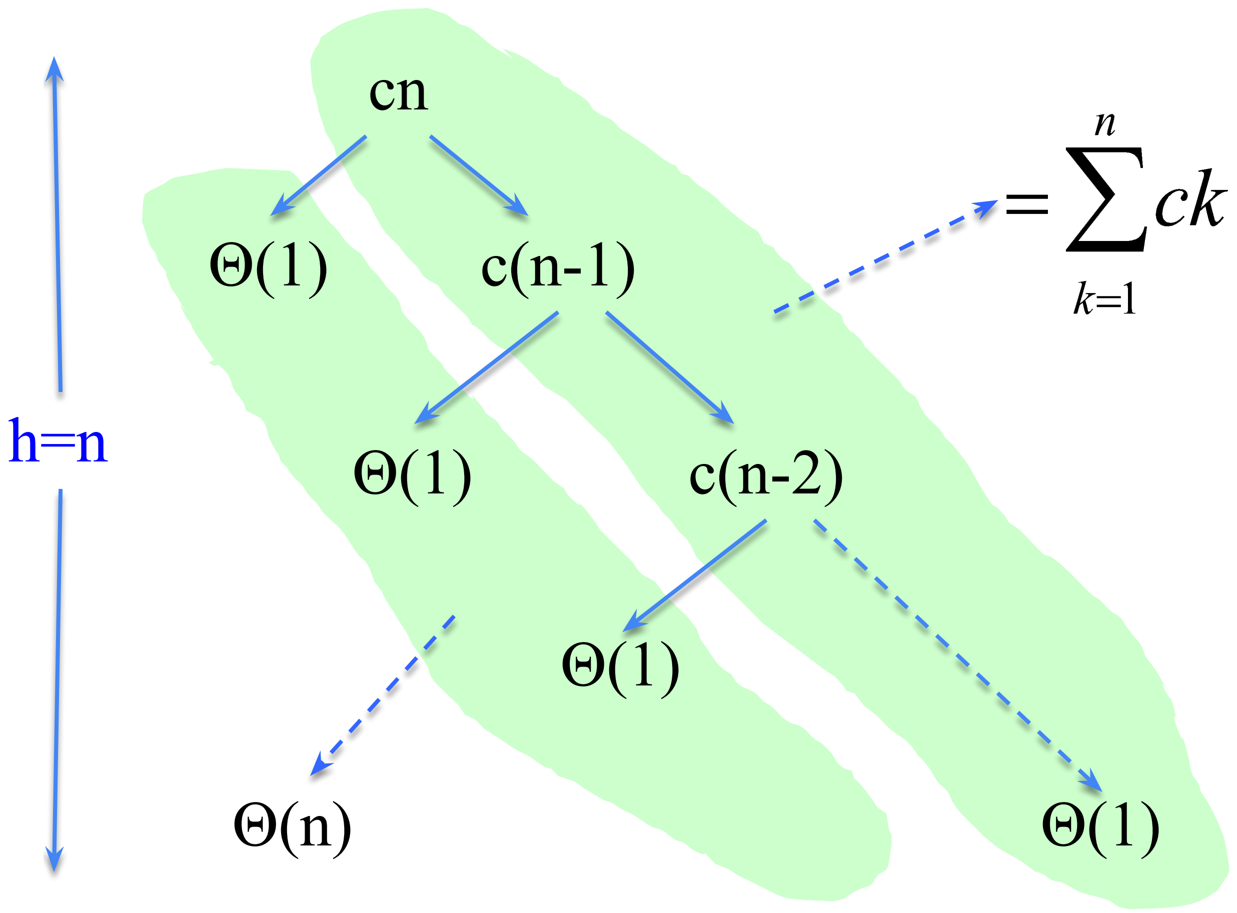
Worst Case Recursion Tree

T(n) = T(1) + T(n-1) + cn

$$\begin{tikzcd}
& cn \arrow[dl] \arrow[dr] & \\
T(1) & & T(n-1)
\end{tikzcd}$$

Worst Case Recursion Tree

$T(n) = T(1) + T(n-1) + cn$



$$= \sum_{k=1}^{n} ck = \Theta(n^2)$$

$T(n) = \Theta(n^2) + \Theta(n)$

$T(n) = \Theta(n^2)$

Best Case Analysis (for intuition only)

- If we're *extremely lucky*, H-PARTITION splits the array *evenly* at *every* recursive call

  $T(n) = 2\,T(n/2) + \Theta(n)$

  $= \Theta(n\lg n)$ □ same as merge sort

- Instead of splitting 0.5:0.5, what if every split is 0.1:0.9?

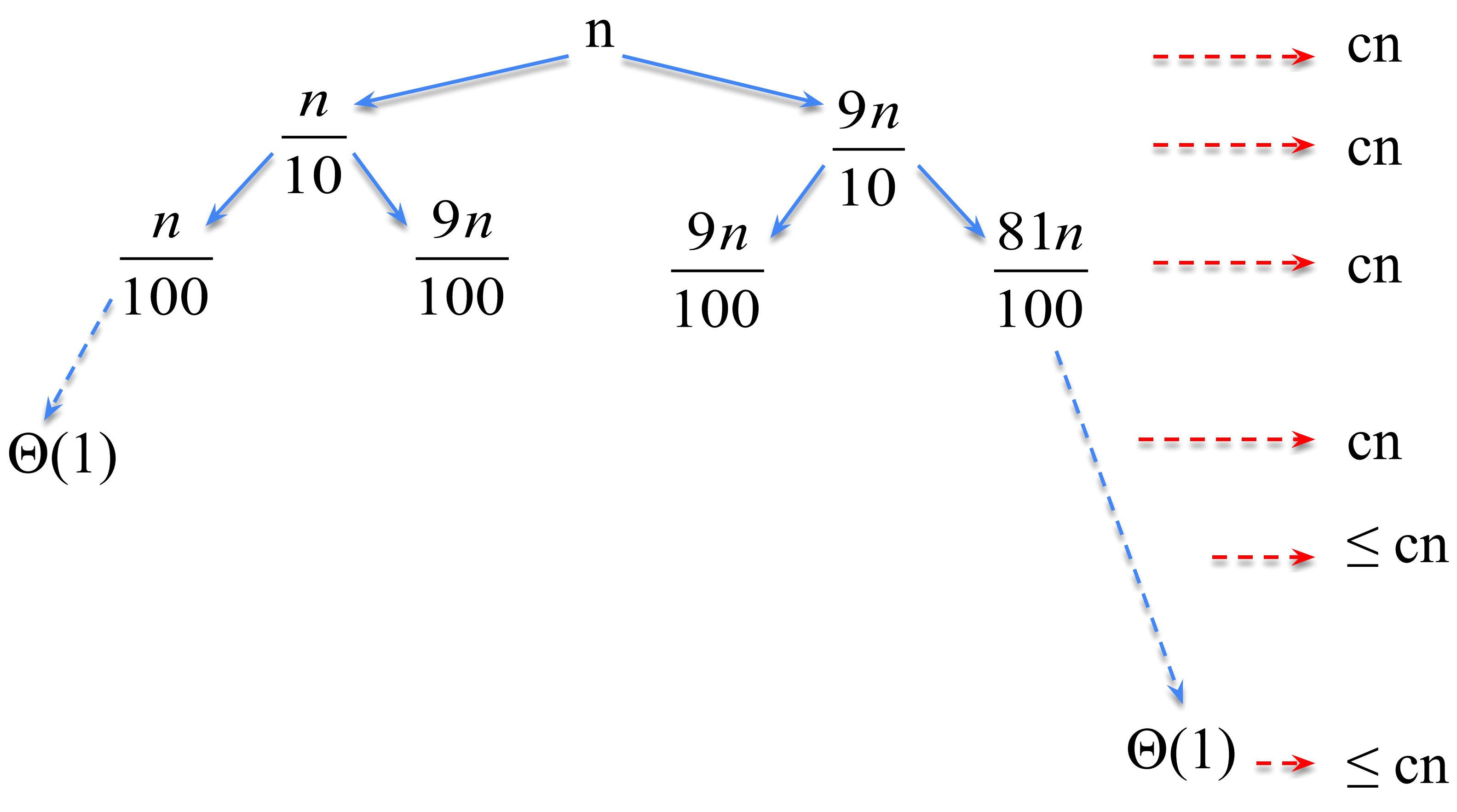
  $T(n) = T(n/10) + T\,(9n/10) + \Theta(n)$

  □ solve this recurrence

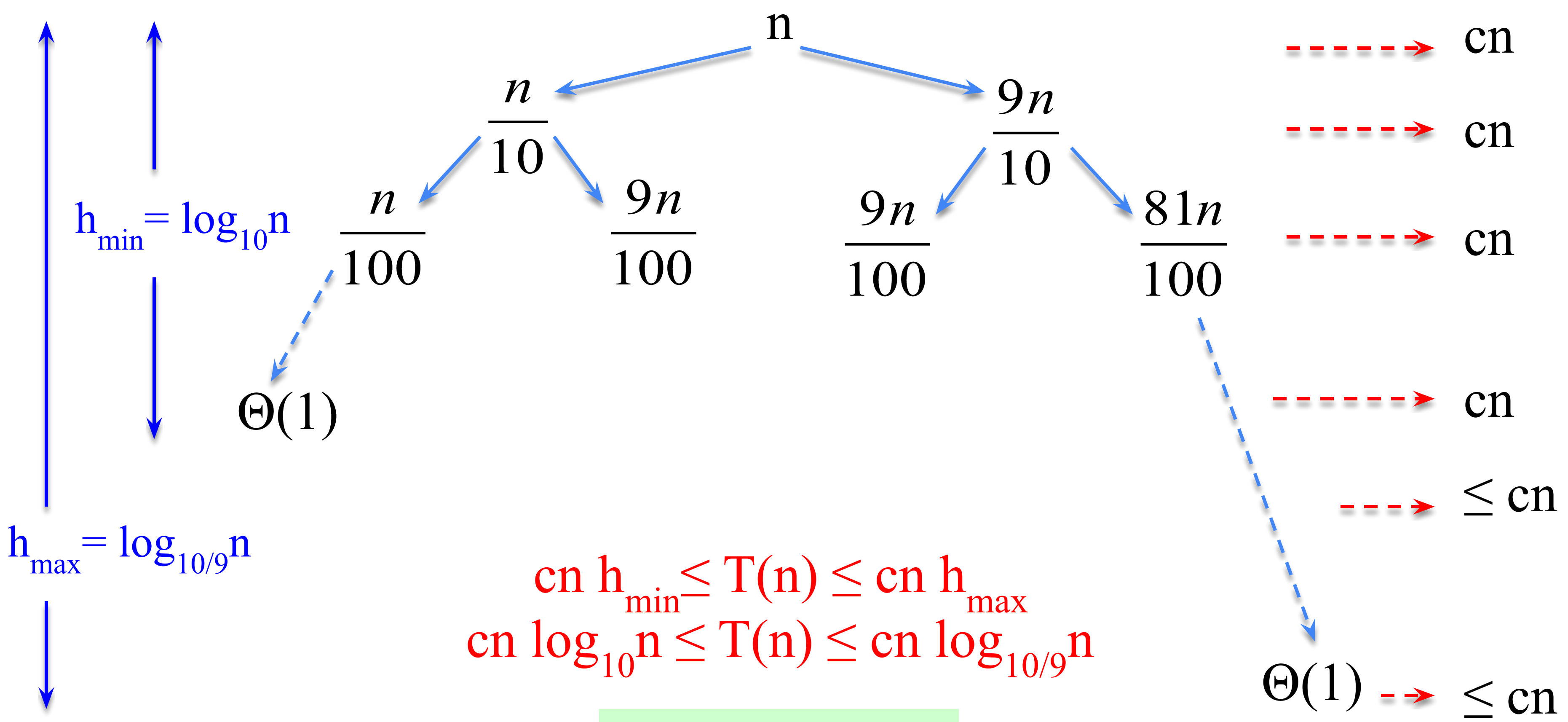
“Almost-Best” Case Analysis

$$\begin{tikzcd}
 & & & n \arrow[dl] \arrow[drr] & & & & \\
 & & \frac{n}{10} \arrow[dl] \arrow[dr] & & & \frac{9n}{10} \arrow[dl] \arrow[dr] & & \\
 & \frac{n}{100} \arrow[dl, dashed] & & \frac{9n}{100} & \frac{9n}{100} & & \frac{81n}{100} \arrow[dr, dashed] & \\
\Theta(1) & & & & & & & \Theta(1)
\end{tikzcd}$$

“Almost-Best” Case Analysis



"Almost-Best" Case Analysis



$$cn\ h_{min} \leq T(n) \leq cn\ h_{max}$$

$$cn \log_{10} n \leq T(n) \leq cn \log_{10/9} n$$

$$T(n) = \Theta(n\lg n)$$

Balanced Partitioning

- We have seen that if H-PARTITION always splits the array with 0.1-to-0.9 ratio, the runtime will be $\Theta(n\lg n)$.
- Same is true with a split ratio of 0.01-to-0.99, etc.
- Possible to show that if the split has always constant ($\Theta(1)$) proportionality, then the runtime will be $\Theta(n\lg n)$.
- In other words, for a *constant* α ($0 < \alpha \le 0.5$):

  α–to–$(1-\alpha)$ proportional split yields $\Theta(n\lg n)$ total runtime

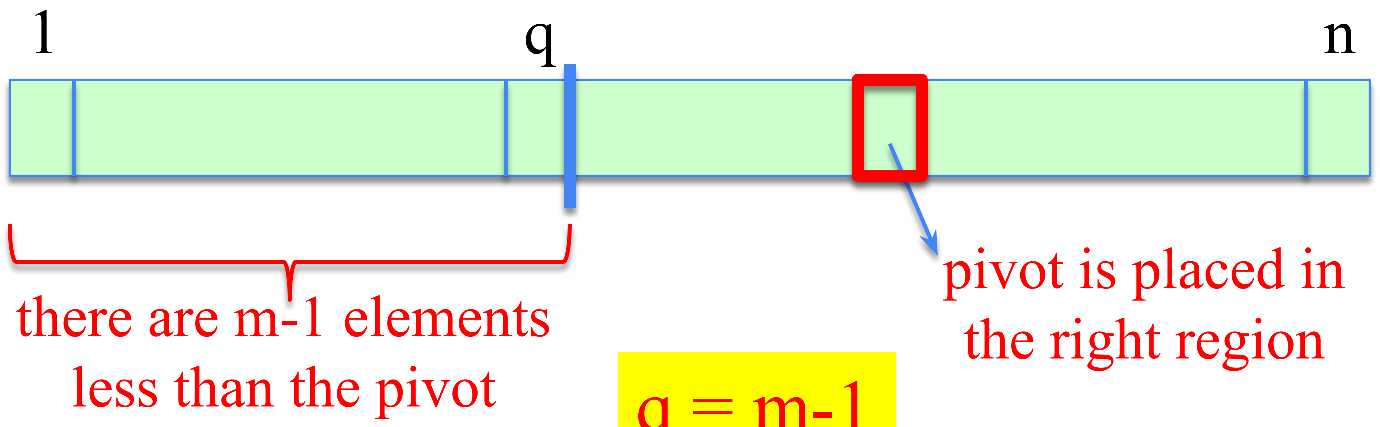
Balanced Partitioning

- In the rest of the analysis, assume that *all input permutations* are equally likely.
    - This is only to gain some intuition
    - We cannot make this assumption for average case analysis
    - We will revisit this assumption later
- Also, assume that all input elements are distinct.
- What is the probability that H-PARTITION returns a split that is more balanced than 0.1-to-0.9?

Balanced Partitioning

Reminder: *H-PARTITION* will place the pivot in the right partition unless the pivot is the smallest element in the arrays.

Question: If the pivot selected is the mth smallest value $(1 < m \leq n)$ in the input array, what is the size of the left region after partitioning?


1
q
n
there are m-1 elements less than the pivot
pivot is placed in the right region

$q = m-1$

Balanced Partitioning

Question: What is the probability that the pivot selected is the m^{th} smallest value in the array of size n?

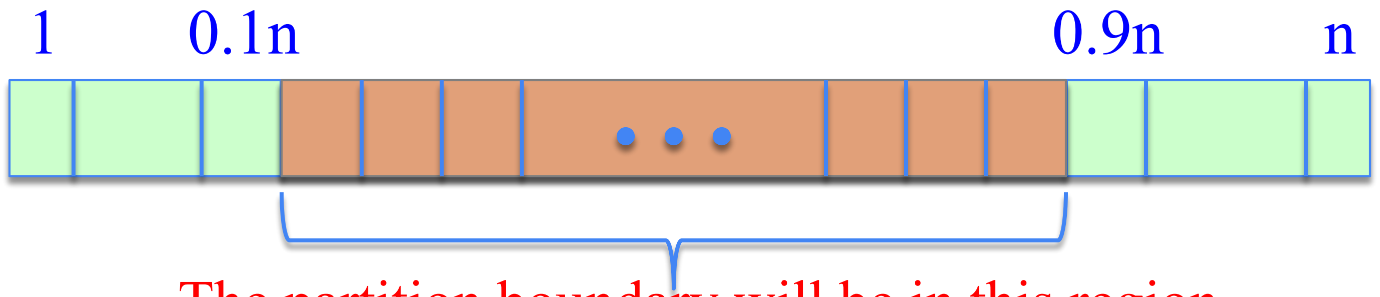
$1/n$ (since all input permutations are equally likely)

Question: What is the probability that the left partition returned by H-PARTITION has size m, where $1 < m < n$?

$1/n$ (due to the answers to the previous 2 questions)

Balanced Partitioning

Question: What is the probability that H-PARTITION returns a split that is more balanced than 0.1-to-0.9?



The partition boundary will be in this region for a more balanced split than 0.1-to-0.9

$$\text{Probability} = \sum_{q=0.1n+1}^{0.9n-1} \frac{1}{n} = \frac{1}{n}(0.9n - 1 - 0.1n - 1 + 1) = 0.8 - \frac{1}{n}$$

≈ 0.8 for large n

Balanced Partitioning

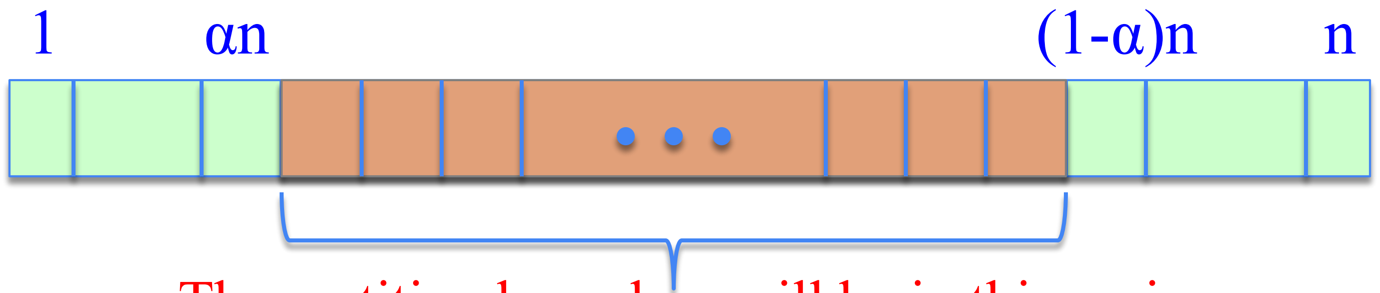
- The probability that *H-PARTITION* yields a split that is more balanced than 0.1-to-0.9 is 80% on a random array.

- Let $P_{\alpha>}$ be the probability that *H-PARTITION* yields a split more balanced than α-to-$(1-\alpha)$, where $0 < \alpha \leq 0.5$

- Repeat the analysis to generalize the previous result

Balanced Partitioning

<u>Question</u>: What is the probability that H-PARTITION returns a split that is more balanced than α-to-(1-α)?



The partition boundary will be in this region for a more balanced split than αn-to-(1-α)n

$$\text{Probability} = \sum_{q=\alpha n+1}^{(1-\alpha)n-1} \frac{1}{n} = \frac{1}{n}((1-\alpha)n-1-\alpha n-1+1) = (1-2\alpha)-\frac{1}{n}$$

≈ (1-2α) for large n

Balanced Partitioning

- We found $P_{\alpha>} = 1 - 2\alpha$

  Examples: $P_{0.1>} = 0.8$ $P_{0.01>} = 0.98$

- Hence, *H-PARTITION* produces a split
  - *more balanced* than a
    - 0.1-to-0.9 split 80% of the time
    - 0.01-to-0.99 split 98% of the time
  - less balanced than a
    - 0.1-to-0.9 split 20% of the time
    - 0.01-to-0.99 split 2% of the time

Intuition for the Average Case

- Assumption: All permutations are equally likely
  - Only for intuition; we'll revisit this assumption later
- Unlikely: Splits always the same way at every level

- Expectation:

  Some splits will be reasonably balanced

  Some splits will be fairly unbalanced
- Average case: A mix of good and bad splits

  Good and *bad* splits distributed randomly thru the tree

Intuition for the Average Case

- *Assume for intuition*: Good and bad splits occur in the alternate levels of the tree

  Good split: Best case split

  Bad split: Worst case split

Intuition for the Average Case

Compare 2-successive levels of avg case vs. 1 level of best case

AVERAGE CASE

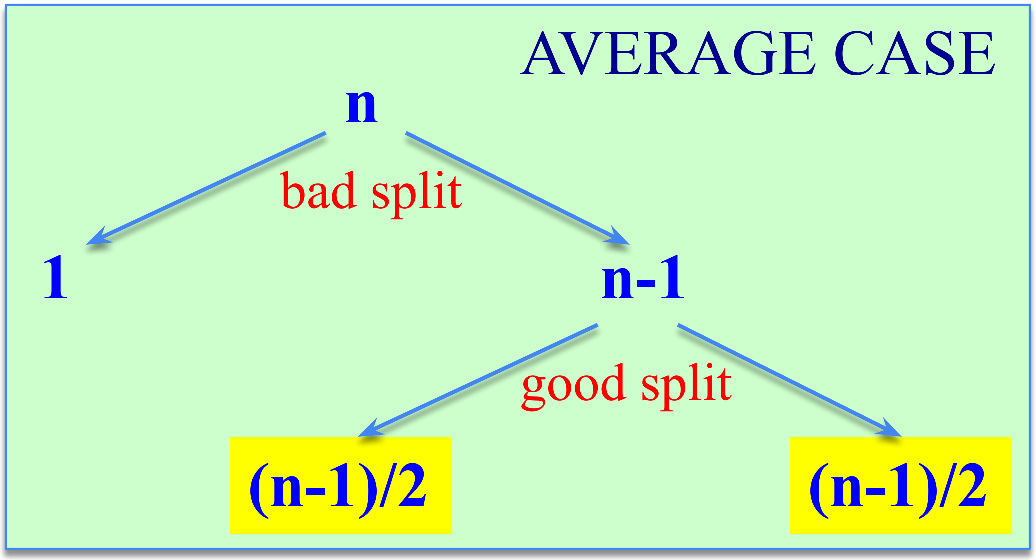
$$\begin{tikzcd}
 & \mathbf{n} \arrow[dl] \arrow[dr, "\text{bad split}"'] & & \\
\mathbf{1} & & \mathbf{n-1} \arrow[dl] \arrow[dr, "\text{good split}"'] & \\
 & \mathbf{(n-1)/2} & & \mathbf{(n-1)/2}
\end{tikzcd}$$

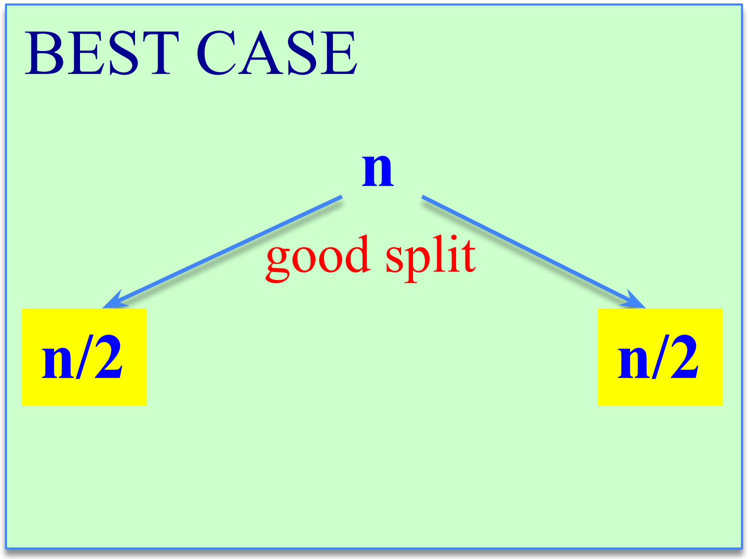
BEST CASE

$$\begin{tikzcd}
 & \mathbf{n} \arrow[dl] \arrow[dr, "\text{good split}"'] & \\
\mathbf{n/2} & & \mathbf{n/2}
\end{tikzcd}$$

Intuition for the Average Case

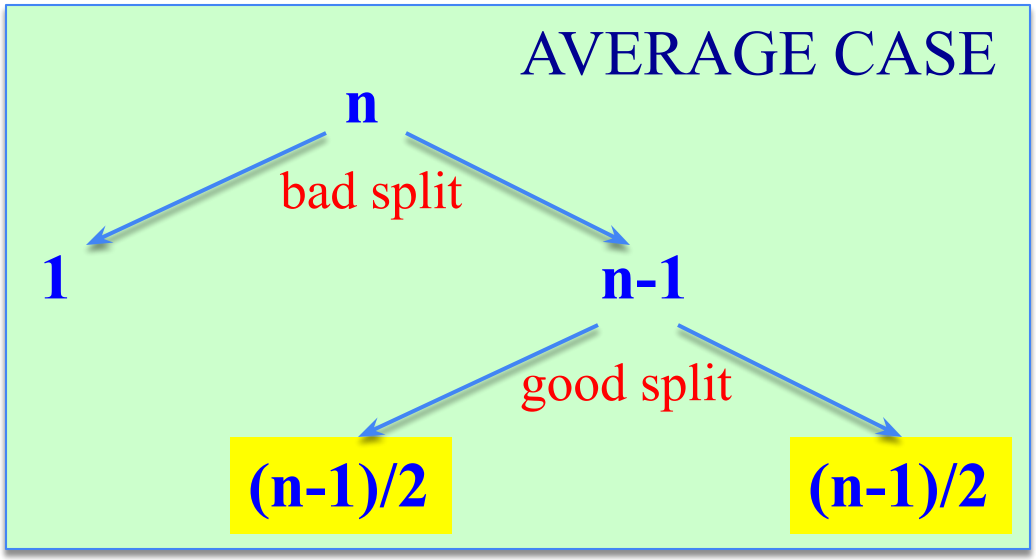
- In terms of the remaining subproblems, two levels of avg case is slightly better than the single level of the best case
- The avg case has extra divide cost of $\Theta(n)$ at alternate levels

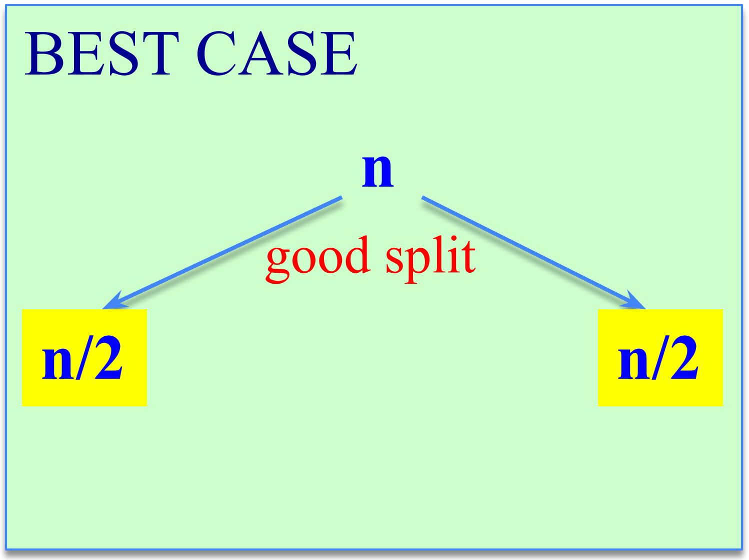




Intuition for the Average Case

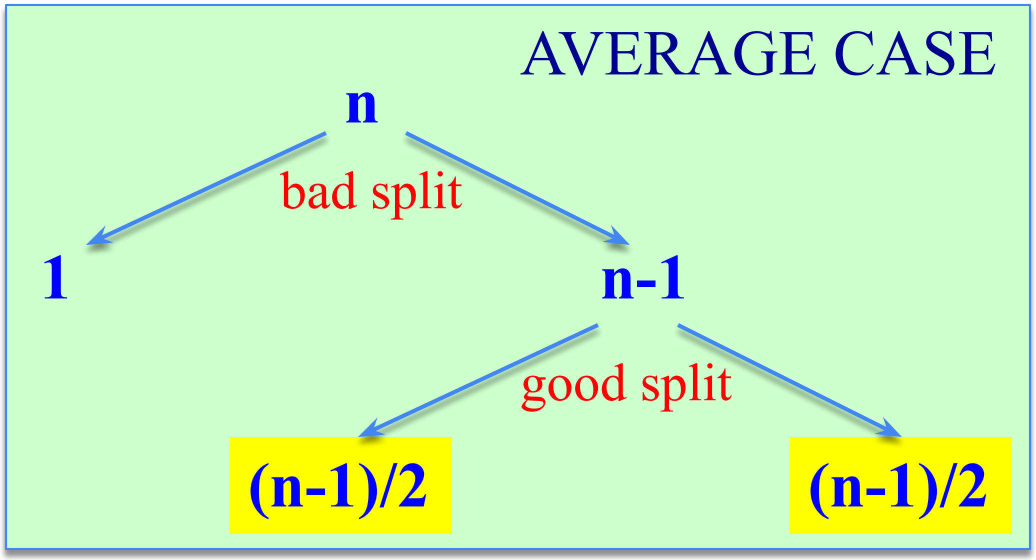
- The extra divide cost Θ(n) of bad splits absorbed into the Θ(n) of good splits.
- Running time is still Θ(nlgn)

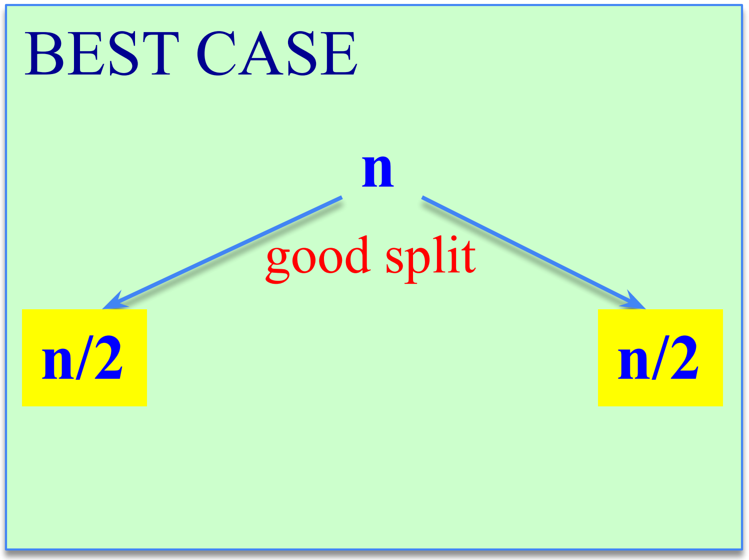




Intuition for the Average Case

- Running time is still $\Theta(n\lg n)$
  - But, slightly larger hidden constants, because the height of the recursion tree is about twice of that of best case.





Intuition for the Average Case

- Another way of looking at it:

  Suppose we alternate lucky, unlucky, lucky, unlucky, …

  We can write the recurrence as:

  $L(n) = 2\,U(n/2) + \Theta(n)$ lucky split (best)

  $U(n) = L(n-1) + \Theta(n)$ unlucky split (worst)

  Solving:

  $L(n) = 2\,(L(n/2-1) + \Theta(n/2)) + \Theta(n)$

  $\quad = 2L(n/2-1) + \Theta(n)$

  $\quad = \Theta(n\lg n)$

  How can we make sure we are usually lucky for all inputs?

Summary: Quicksort Runtime Analysis

Worst case: Unbalanced split at every recursive call

$$T(n) = T(1) + T(n-1) + \Theta(n)$$

□ $T(n) = \Theta(n^2)$

Best case: Balanced split at every recursive call (extremely lucky)

$$T(n) = 2T(n/2) + \Theta(n)$$

□ $T(n) = \Theta(n\lg n)$

Summary: Quicksort Runtime Analysis

Almost-best case: Almost-balanced split at every recursive call

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

or $$T(n) = T(n/100) + T(99n/100) + \Theta(n)$$

or $$T(n) = T(\alpha n) + T((1-\alpha)n) + \Theta(n)$$

for any constant α, $0 < \alpha \leq 0.5$

□ $T(n) = \Theta(n\lg n)$

Summary: Quicksort Runtime Analysis

For a random input array, the probability of having a split

more balanced than $0.1 - \text{to} - 0.9$: 80%

more balanced than $0.01 - \text{to} - 0.99$: 98%

more balanced than $\alpha - \text{to} - (1\text{-}\alpha) : 1 - 2\alpha$

for any constant α, $0 < \alpha \leq 0.5$

Summary: Quicksort Runtime Analysis

<u>Avg case intuition</u>: Different splits expected at different levels

□ some balanced (good), some unbalanced (bad)

Assume the good and bad splits alternate

i.e. good split □ bad split □ good split □ …

□ $T(n) = \Theta(n\lg n)$

(informal analysis for intuition)