CS473 - Algorithms I

Lecture 1

Introduction to Analysis of Algorithms

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Grading

- Midterm: 24%
- Final: 30%
- Classwork: 40%
- Attendance: 6%

Classwork (45% of the total grade)

- Like small exams, covering the most recent material

- There will be 4 classwork sessions

- Check webpage for dates
- Mostly weeknights at 17:40

- Open book (clean and unused). No notes. No slides.
- See the syllabus for details.

Algorithm Definition

- Algorithm: A sequence of computational steps that transform the input to the desired output
- Procedure vs. algorithm
  - *An algorithm* ***must halt within finite time*** *with the right output*
- Example:

Many Real World Applications

- Bioinformatics
  - Determine/compare DNA sequences
- Internet
  - Manage/manipulate/route data
- Information retrieval
  - Search and access information in large data
- Security
  - Encode & decode personal/financial/confidential data
- Electronic design automation
  - Minimize human effort in chip-design process

Course Objectives

- Learn basic algorithms & data structures
- Gain skills to design new algorithms

- Focus on efficient algorithms

- Design algorithms that
  - are fast
  - use as little memory as possible
  - are correct!

Outline of Lecture 1

- Study two sorting algorithms as examples
  - Insertion sort: *Incremental* algorithm
  - Merge sort: *Divide-and-conquer*

- Introduction to runtime analysis
  - Best vs. worst vs. average case
  - Asymptotic analysis

Sorting Problem

Input: Sequence of numbers

$$\langle a_1, a_2, \ldots, a_n \rangle$$

Output: A permutation

$$\Pi = \langle \Pi(1), \Pi(2), \ldots, \Pi(n) \rangle$$

such that

$$a_{\Pi(1)} \leq a_{\Pi(2)} \leq \ldots \leq a_{\Pi(n)}$$

Insertion Sort

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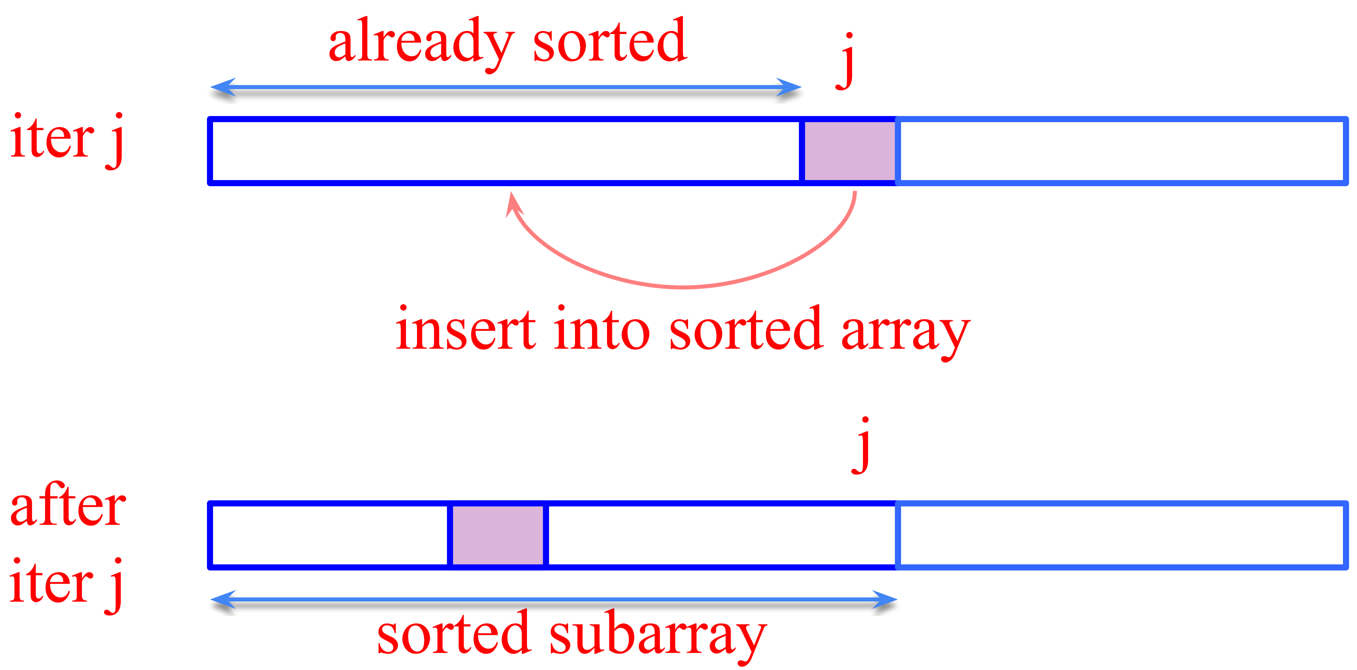
Insertion Sort: Basic Idea

- Assume input array: A[1..n]
- Iterate j from 2 to n



Pseudo-code notation

- Objective: Express algorithms to humans in a clear and concise way

- Liberal use of English

- Indentation for block structures

- Omission of error handling and other details
  - *needed in real programs*

Algorithm: Insertion Sort (from Section 2.2)

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```

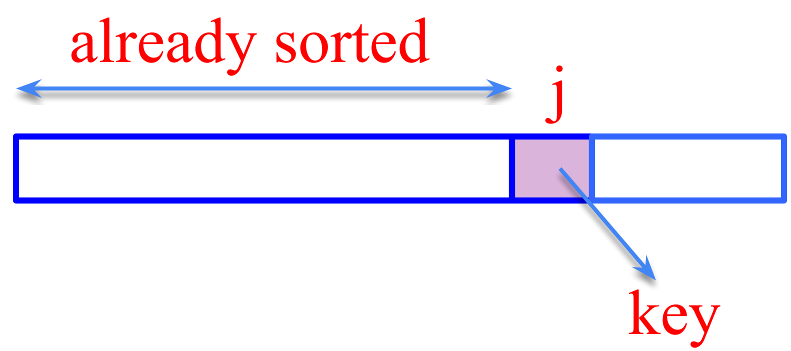
Algorithm: Insertion Sort

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```

Iterate over array elts j

Loop invariant:
The subarray A[1..j-1] is always sorted



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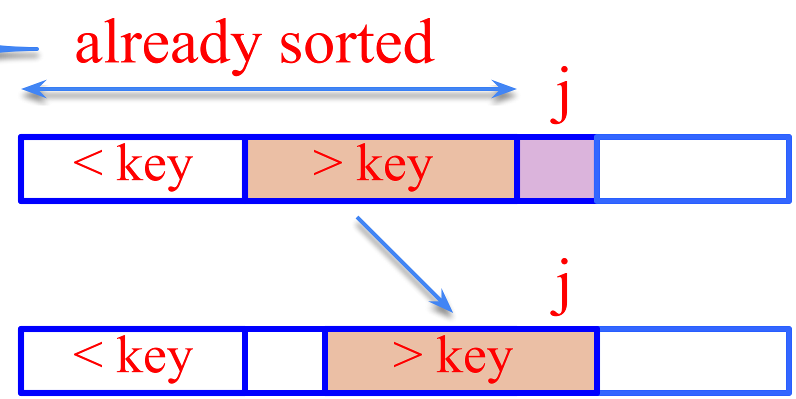
Algorithm: Insertion Sort

<u>Insertion-Sort</u> (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```

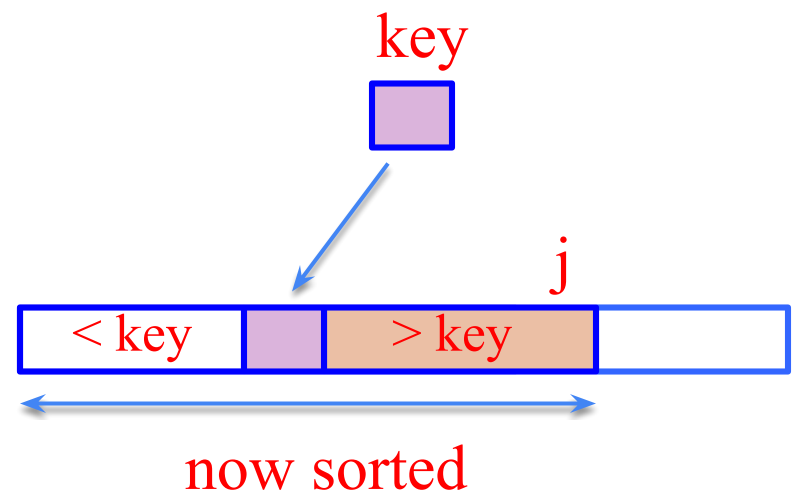
Shift right the entries in A[1..j-1] that are > key



Algorithm: Insertion Sort

Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```



Insert key to the correct location

End of iter j: A[1..j] is sorted

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Insertion Sort - Example

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.       key ← A[j];
3.       i ← j - 1;
4.       while i > 0 and A[i] > key
         do
5.            A[i+1] ← A[i];
6.            i ← i - 1;
         endwhile
7.       A[i+1] ← key;
    endfor
```

5	**2**	**4**	**6**	**1**	**3**

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Insertion Sort - Example: Iteration j=2

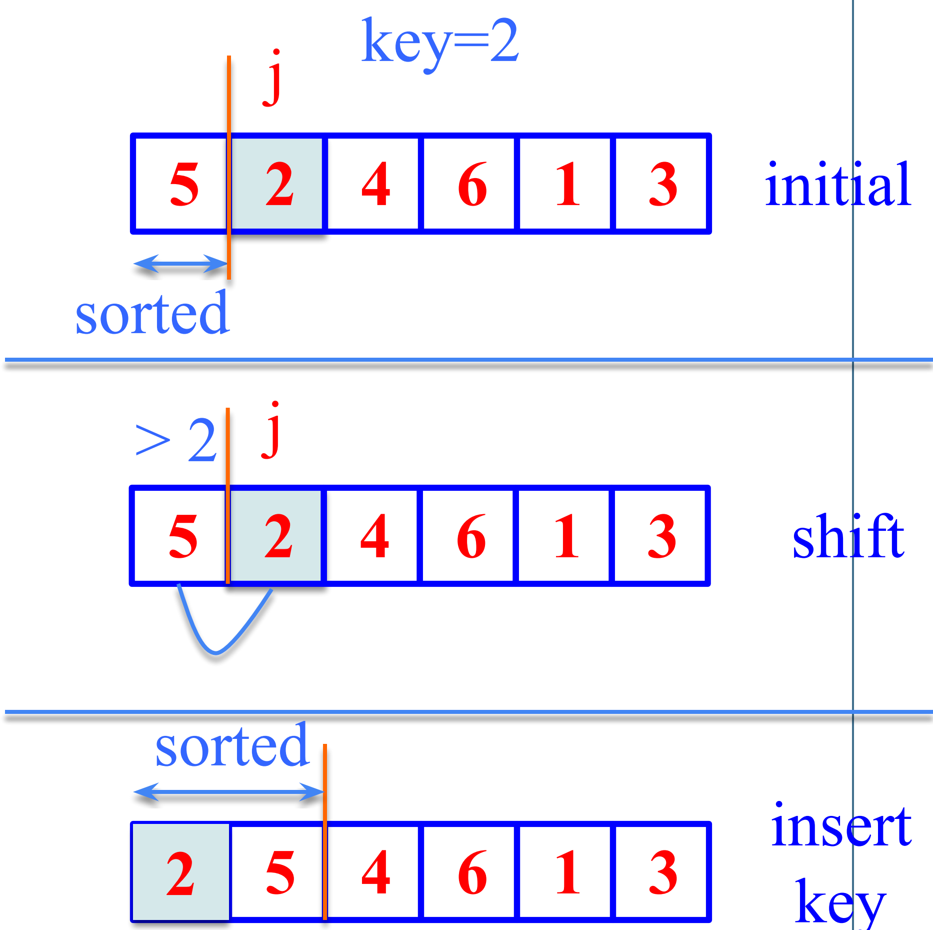
Insertion-Sort (A)

1. **for** j ← 2 to n **do**
2. key ← A[j];
3. i ← j - 1;
4. **while** i > 0 **and** A[i] > key **do**
5. A[i+1] ← A[i];
6. i ← i - 1;

   endwhile

7. A[i+1] ← key;

endfor



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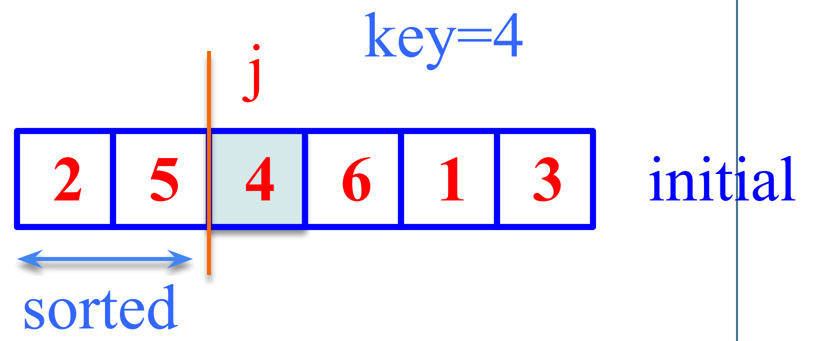
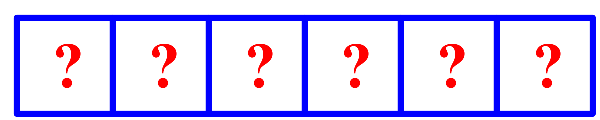
Insertion Sort - Example: Iteration j=3

Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```



What are the entries at the end of iteration j=3?



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Insertion Sort - Example: Iteration j=3

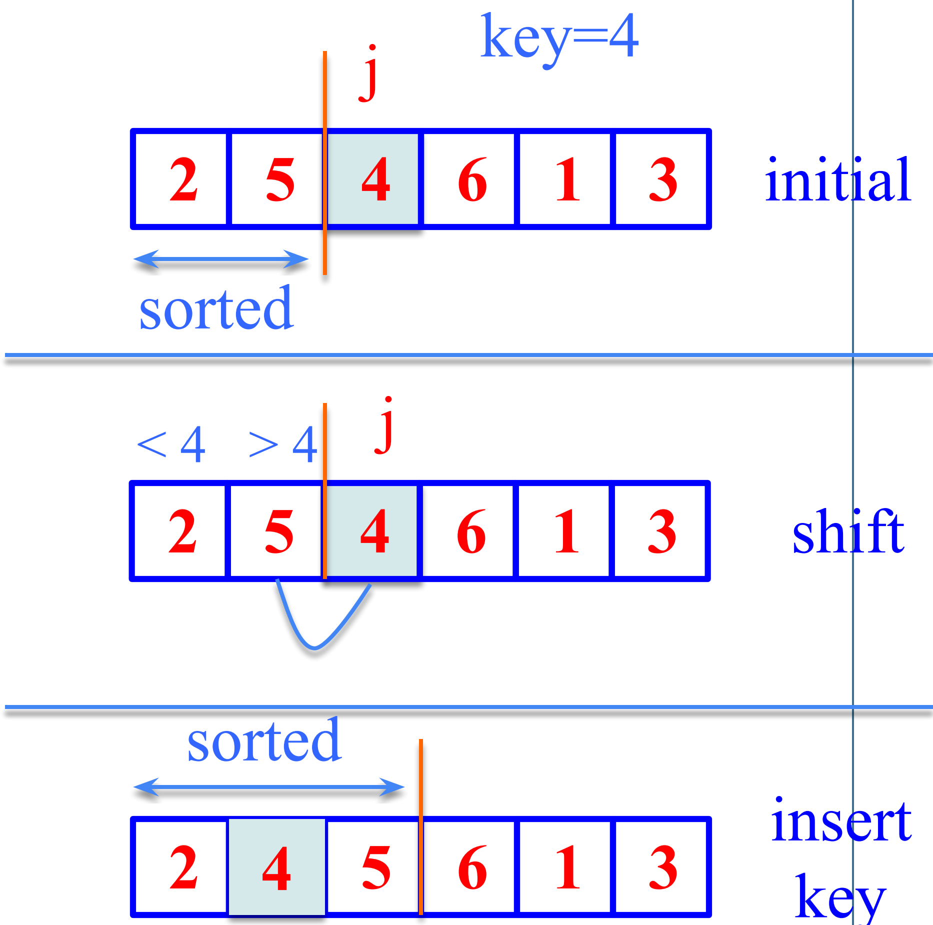
Insertion-Sort (A)

1. **for** j ← 2 **to** n **do**
2. key ← A[j];
3. i ← j - 1;
4. **while** i > 0 **and** A[i] > key **do**
5. A[i+1] ← A[i];
6. i ← i - 1;

   endwhile

7. A[i+1] ← key;

endfor



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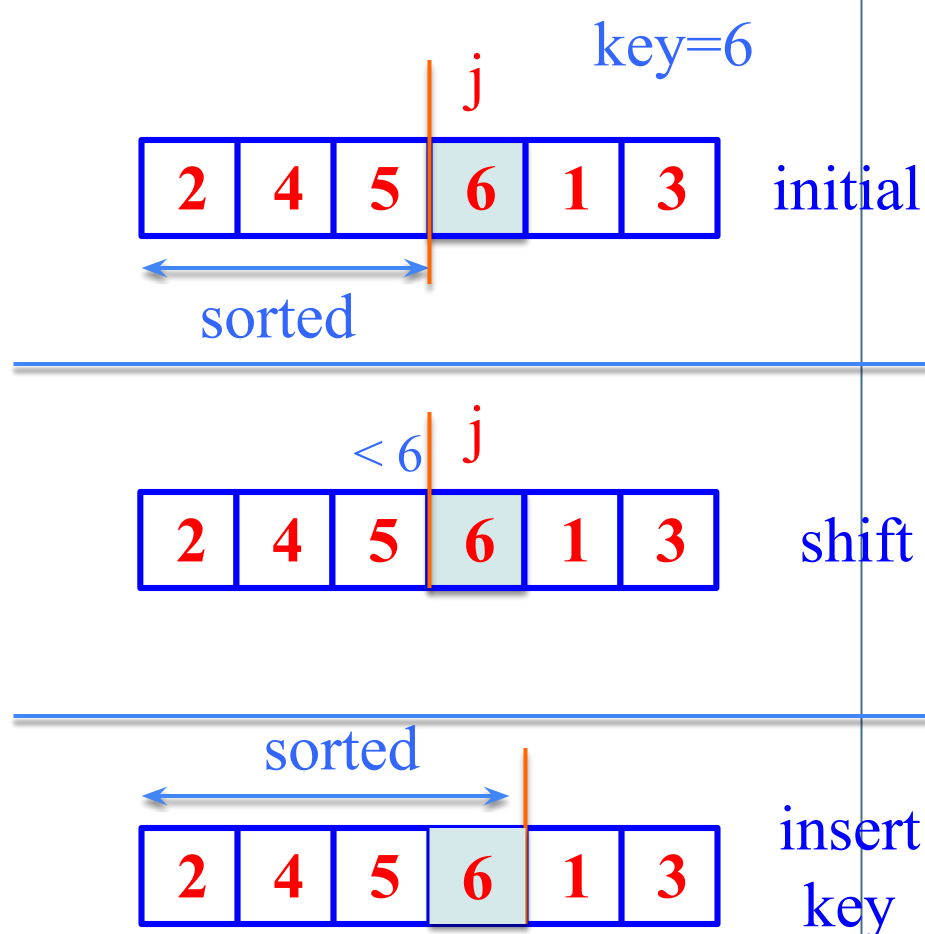
Insertion Sort - Example: Iteration j=4

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



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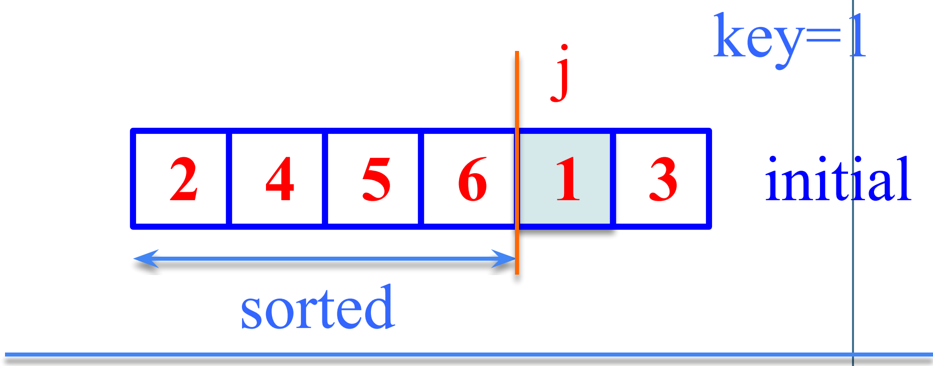
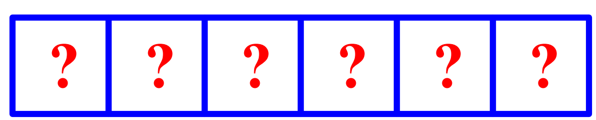
Insertion Sort - Example: Iteration j=5

Insertion-Sort (A)

```
1. for j ← 2 to n do
2.     key ← A[j];
3.     i ← j - 1;
4.     while i > 0 and A[i] > key
       do
5.         A[i+1] ← A[i];
6.         i ← i - 1;
       endwhile
7.     A[i+1] ← key;
   endfor
```



What are the entries at the end of iteration j=5?



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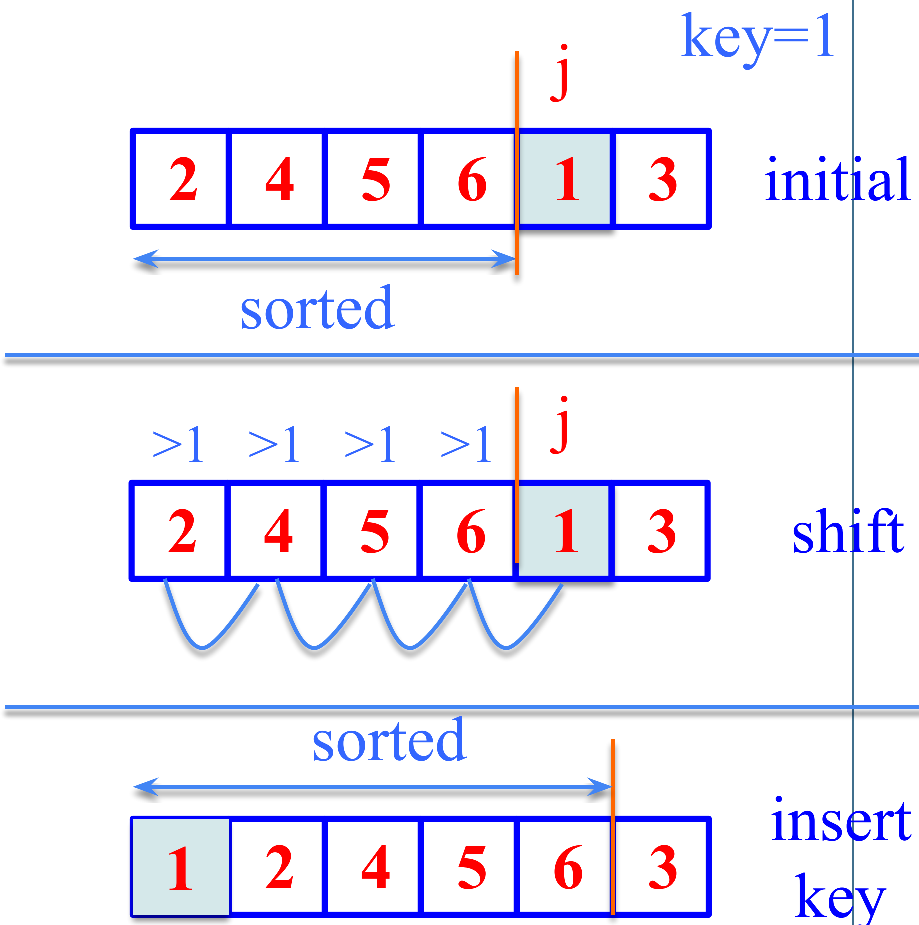
Insertion Sort - Example: Iteration j=5

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



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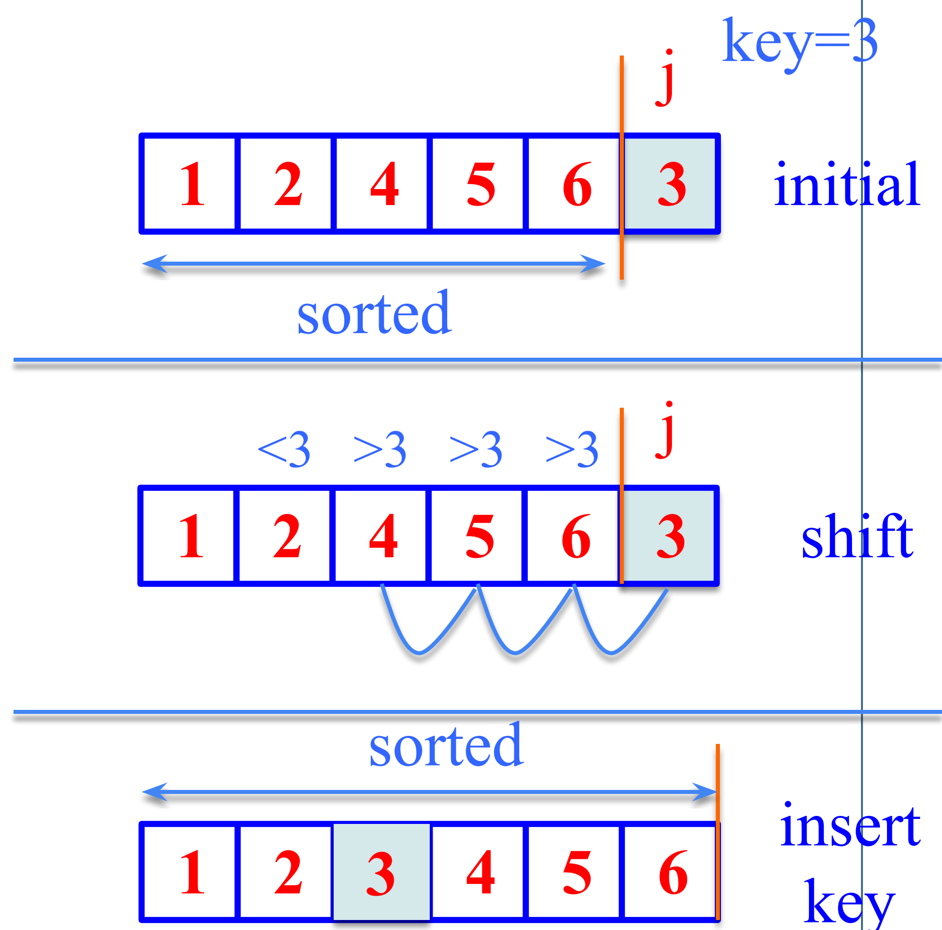
Insertion Sort - Example: Iteration j=6

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



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Insertion Sort Algorithm - Notes

- Items sorted in-place
  - Elements rearranged within array
  - At most constant number of items stored outside the array at any time (e.g. the variable *key*)
  - Input array A contains sorted output sequence when the algorithm ends

- Incremental approach
  - Having sorted *A[1..j-1]*, place *A[j]* correctly so that *A[1..j]* is sorted

Running Time

- Depends on:
  - Input size (e.g., 6 elements vs 6M elements)
  - Input itself (e.g., partially sorted)

- Usually want *upper bound*

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Kinds of running time analysis

- Worst Case *(Usually)*

  $T(n)$ = max time on any input of size n

- Average Case (*Sometimes*)

  $T(n)$ = average time over all inputs of size n

  Assumes statistical distribution of inputs

- Best Case (*Rarely*)

  $T(n)$ = min time on any input of size n

  BAD*: Cheat with slow algorithm that works fast on some inputs

  GOOD: Only for showing bad lower bound

* Can modify any algorithm (almost) to have a low best-case running time
  - Check whether input constitutes an output at the very beginning of the algorithm

Running Time

- For Insertion-Sort, what is its worst-case time?
  - Depends on speed of primitive operations
    - Relative speed (on same machine)
    - Absolute speed (on different machines)

- Asymptotic analysis
  - Ignore machine-dependent constants
  - Look at growth of $T(n)$ as $n \to \infty$

Θ Notation

- Drop low order terms
- Ignore leading constants

e.g.

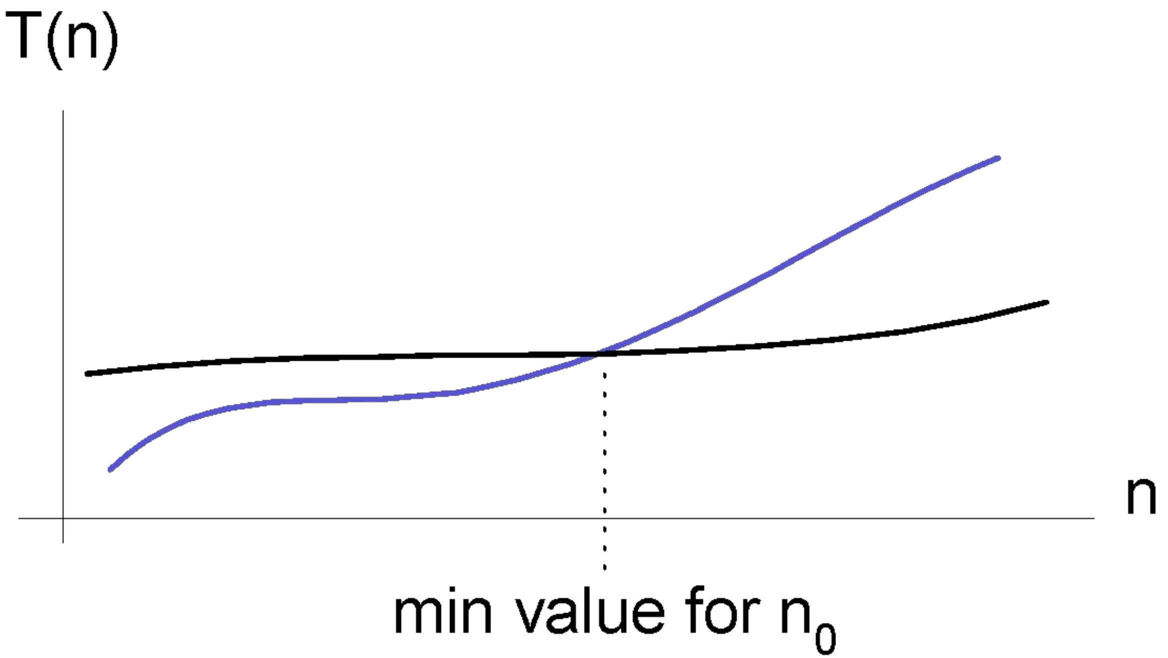
$$2\mathbf{n}^2 + 5n + 3 = \Theta(n^2)$$

$$3\mathbf{n}^3 + 90n^2 - 2n + 5 = \Theta(n^3)$$

❑ *Formal explanations in the next lecture.*

Θ Notation

- As n gets large, a $\Theta(n^2)$ algorithm runs faster than a $\Theta(n^3)$ algorithm



Insertion Sort – Runtime Analysis

Cost

c_1 - - - - - - - -
c_2 - - - - - - - -
c_3 - - - - - - - -
c_4 - - - - - - - -
c_5 - - - - - - - -
c_6 - - - - - - - -
c_7 - - - - - - - -

Insertion-Sort (A)

```
1. for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.           A[i+1] ← A[i];
6.           i ← i - 1;
        endwhile
7.      A[i+1] ← key;
     endfor
```

t_j: The number of times while loop test is executed for j

How many times is each line executed?

# times	Insertion-Sort (A)
	1. **for** j ← 2 **to** n **do**
n	2. key ← A[j];
n-1	3. i ← j - 1;
n-1	4. **while** i > 0 **and** A[i] > key
k_4	**do**
	5. A[i+1] ← A[i];
k_5	6. i ← i - 1;
k_6	**endwhile**
	7. A[i+1] ← key;
n-1	**endfor**

$$k_4 = \sum_{j=2}^{n} t_j$$

$$k_5 = \sum_{j=2}^{n} (t_j - 1)$$

$$k_6 = \sum_{j=2}^{n} (t_j - 1)$$

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Insertion Sort – Runtime Analysis

- Sum up costs:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

- What is the best case runtime?

- What is the worst case runtime?

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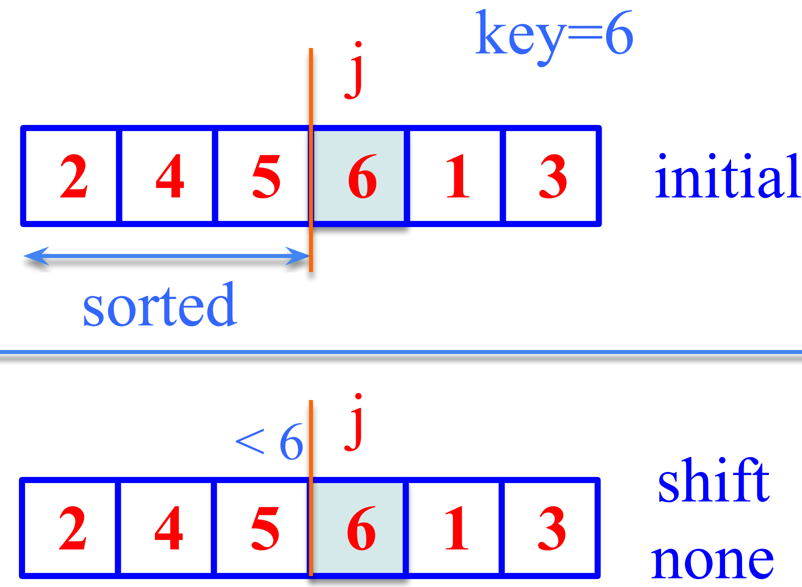
Question: If A[1...j] is already sorted, t_j = ?

```
Insertion-Sort (A)
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```



$$t_j = 1$$

Insertion Sort – Best Case Runtime

- Original function:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

- Best-case: Input array is already sorted

  $t_j = 1$ for all j

$$T(n) = (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

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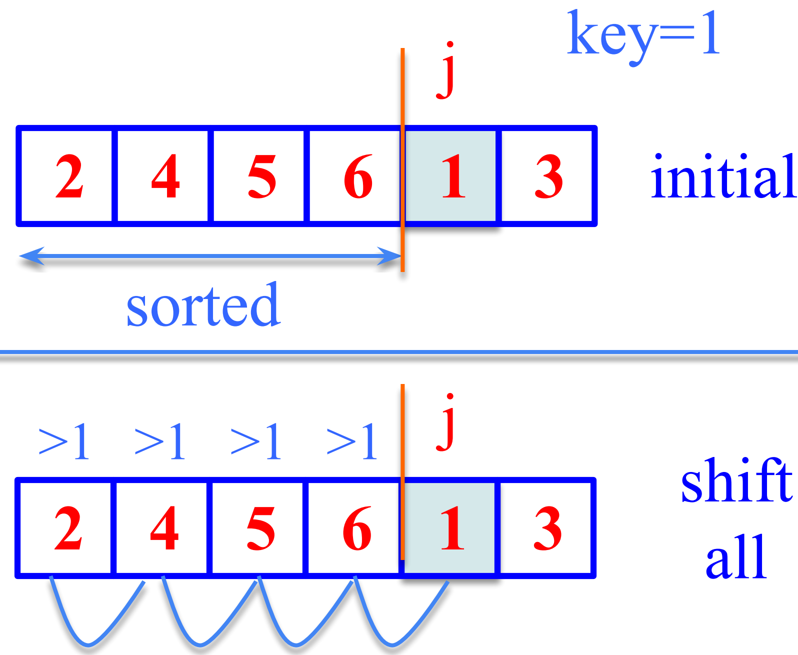
Q: If A[j] is smaller than every entry in A[1..j-1], $t_j = ?$

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



$$t_j = j$$

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Insertion Sort – Worst Case Runtime

- Worst case: The input array is reverse sorted

  $t_j = j$ for all j

- After derivation, worst case runtime:

$$T(n) = \tfrac{1}{2}(c_4 + c_5 + c_6)n^2 + (c_1 + c_2 + c_3 + \tfrac{1}{2}(c_4 - c_5 - c_6) + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

Insertion Sort – Asymptotic Runtime Analysis

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];             } Θ(1)
3.      i ← j - 1;              }
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];      } Θ(1)
6.          i ← i - 1;          }
        endwhile
7.      A[i+1] ← key;           } Θ(1)
    endfor
```


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Asymptotic Runtime Analysis of Insertion-Sort

- Worst-case (input reverse sorted)
  - *Inner loop is Θ(j)*

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(\sum_{j=2}^{n} j\right) = \Theta(n^2)$$

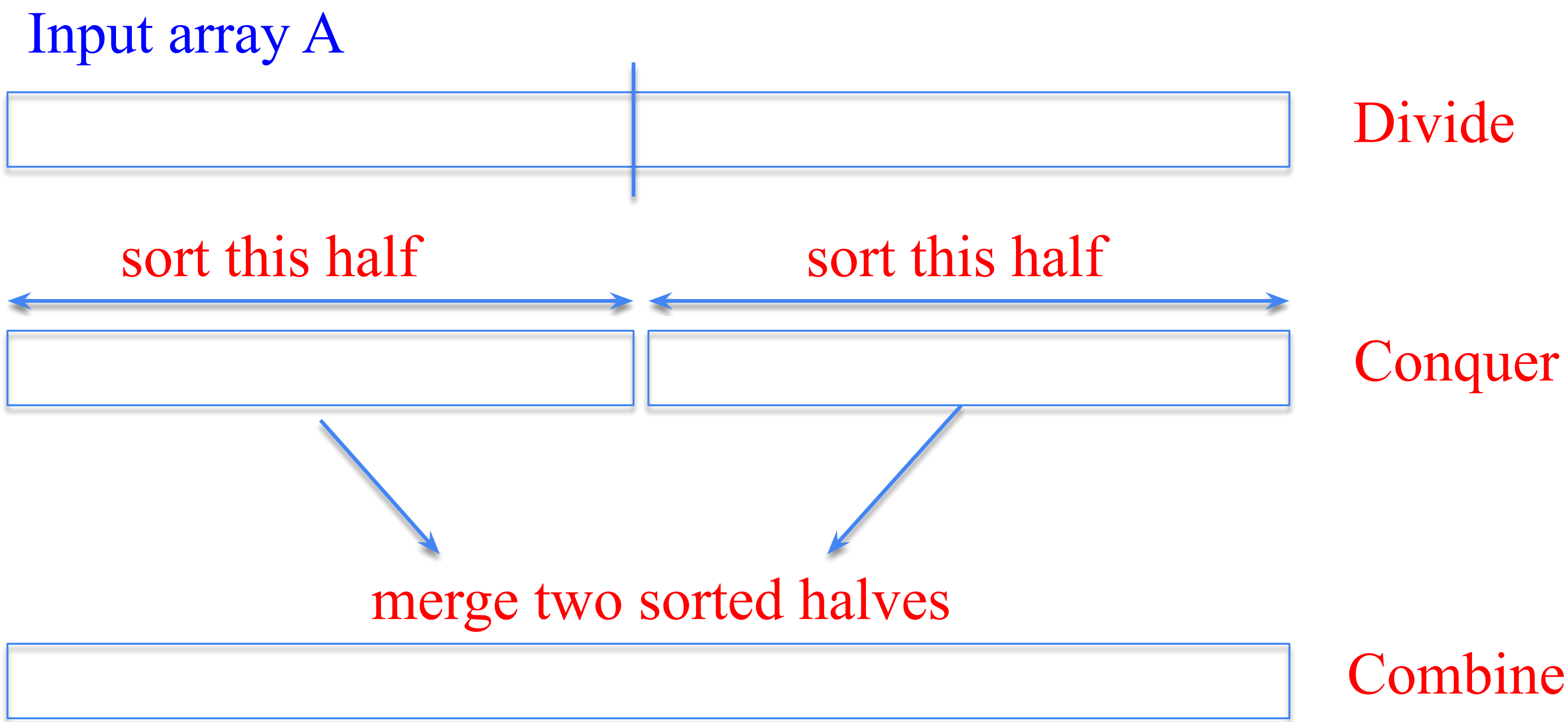
- Average case (all permutations equally likely)
  - *Inner loop is Θ(j/2)*

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

    - Often, average case not much better than worst case

- Is this a fast sorting algorithm?
  - Yes, for small n. No, for large n.

Merge Sort: Basic Idea



Merge-Sort (A, p, r)

```
if p = r then return;
else
    q ← ⌊(p+r)/2⌋;            (Divide)
    Merge-Sort (A, p, q);      (Conquer)
    Merge-Sort (A, q+1, r);    (Conquer)
    Merge (A, p, q, r);        (Combine)
endif
```

- Call Merge-Sort(A,1,n) to sort A[1..n]
- Recursion bottoms out when subsequences have length 1

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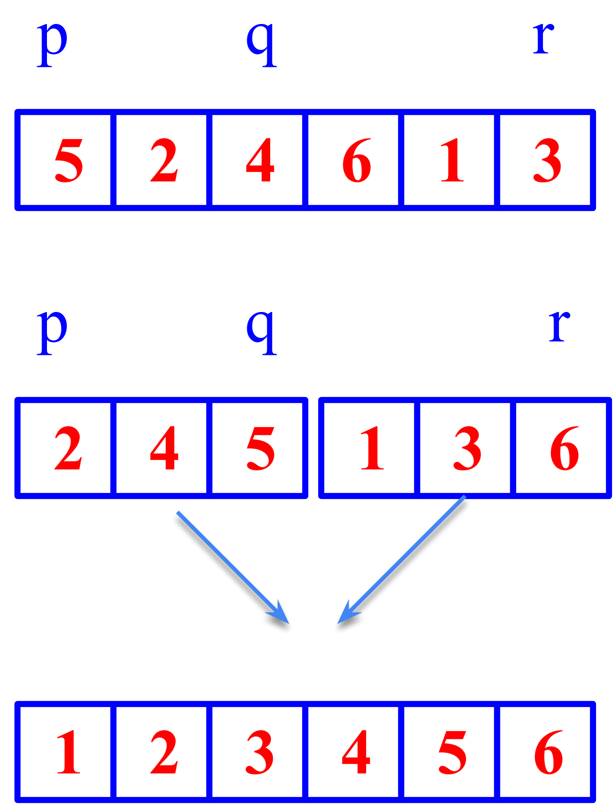
Merge Sort: Example

Merge-Sort (A, p, r)
  if p = r **then**
    return
  else
    q ← $\lfloor (p+r)/2 \rfloor$

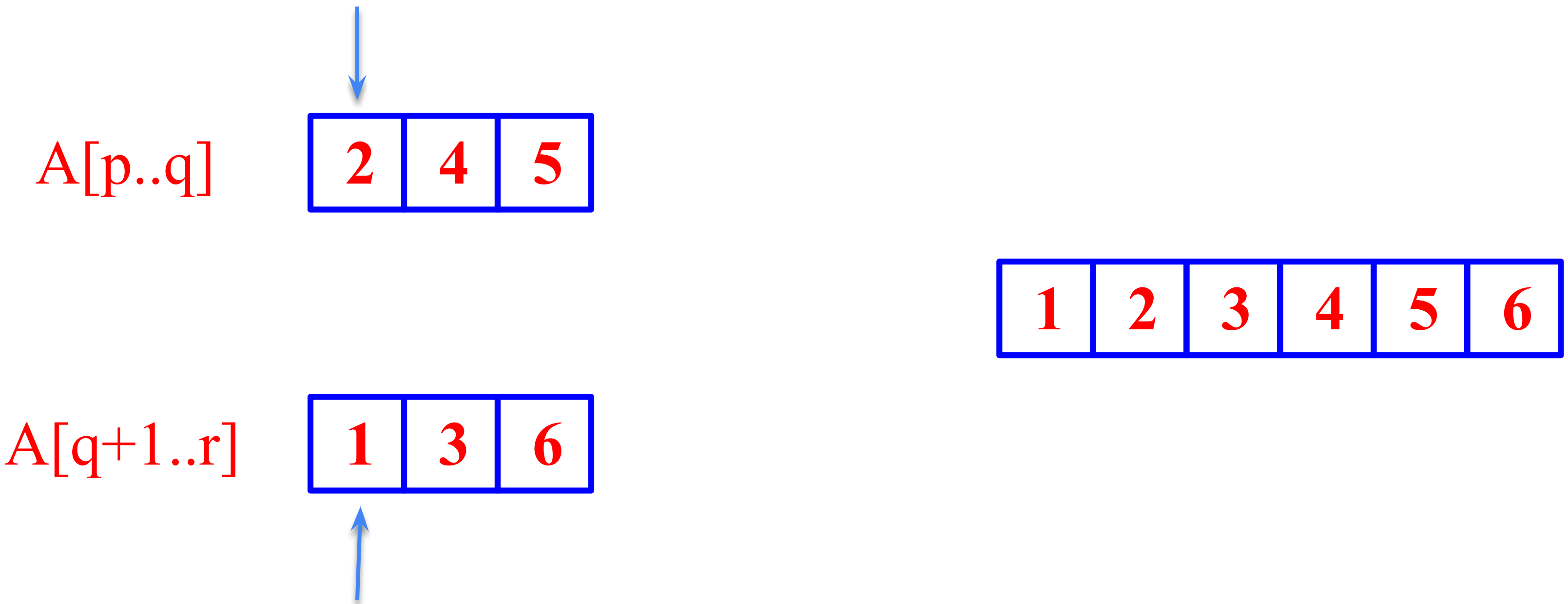
    Merge-Sort (A, p, q)
    Merge-Sort (A, q+1, r)

    Merge(A, p, q, r)
  endif



How to merge 2 sorted subarrays?

- *HW: Study the pseudo-code in the textbook (Sec. 2.3.1)*
- What is the complexity of this step?



$\Theta(n)$

Merge Sort: Correctness

```
Merge-Sort (A, p, r)
  if p = r then
      return
  else
      q ← ⌊ (p+r)/2⌋

      Merge-Sort  (A, p, q)
      Merge-Sort  (A, q+1, r)

      Merge(A, p, q, r)
  endif
```

Base case: $p = r$
 Trivially correct

Inductive hypothesis: MERGE-SORT is correct for any subarray that is a ***strict*** (smaller) ***subset*** of A[p, q].

General Case: MERGE-SORT is correct for A[p, q].
 From inductive hypothesis and correctness of *Merge*.

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Merge Sort: Complexity

Merge-Sort (A, p, r)	→	$T(n)$
if p = r **then** **return**	→	$\Theta(1)$
else		
$q \leftarrow \lfloor (p+r)/2 \rfloor$	→	$\Theta(1)$
Merge-Sort (A, p, q)	→	$T(n/2)$
Merge-Sort (A, q+1, r)	→	$T(n/2)$
Merge(A, p, q, r)	→	$\Theta(n)$
endif		

Merge Sort – Recurrence

- Describe a function recursively in terms of itself
- To analyze the performance of recursive algorithms

- For merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

How to solve for T(n)?

- Generally, we will assume $T(n) = \Theta(1)$ for sufficiently small n

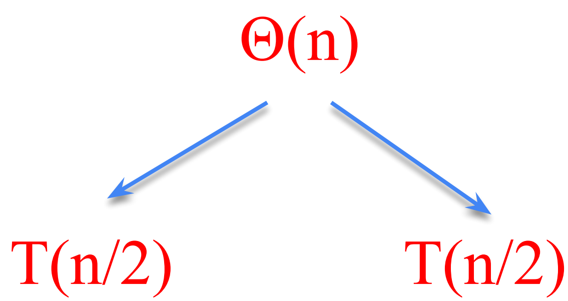
$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

- The recurrence above can be rewritten as:

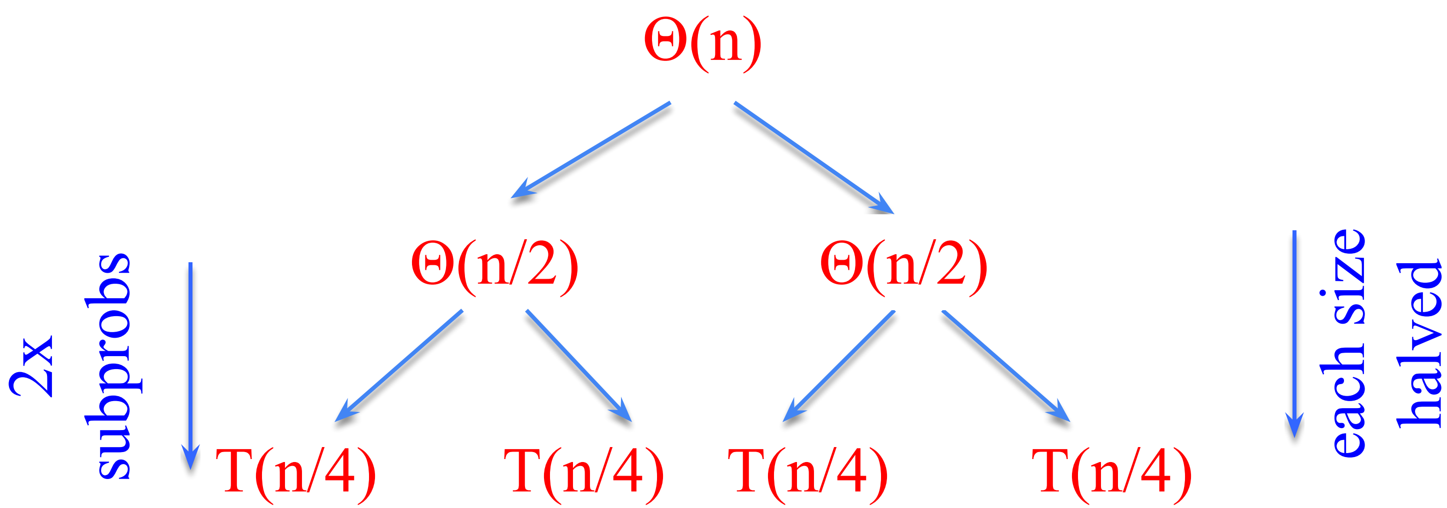
$$T(n) = 2\,T(n/2) + \Theta(n)$$

- How to solve this recurrence?

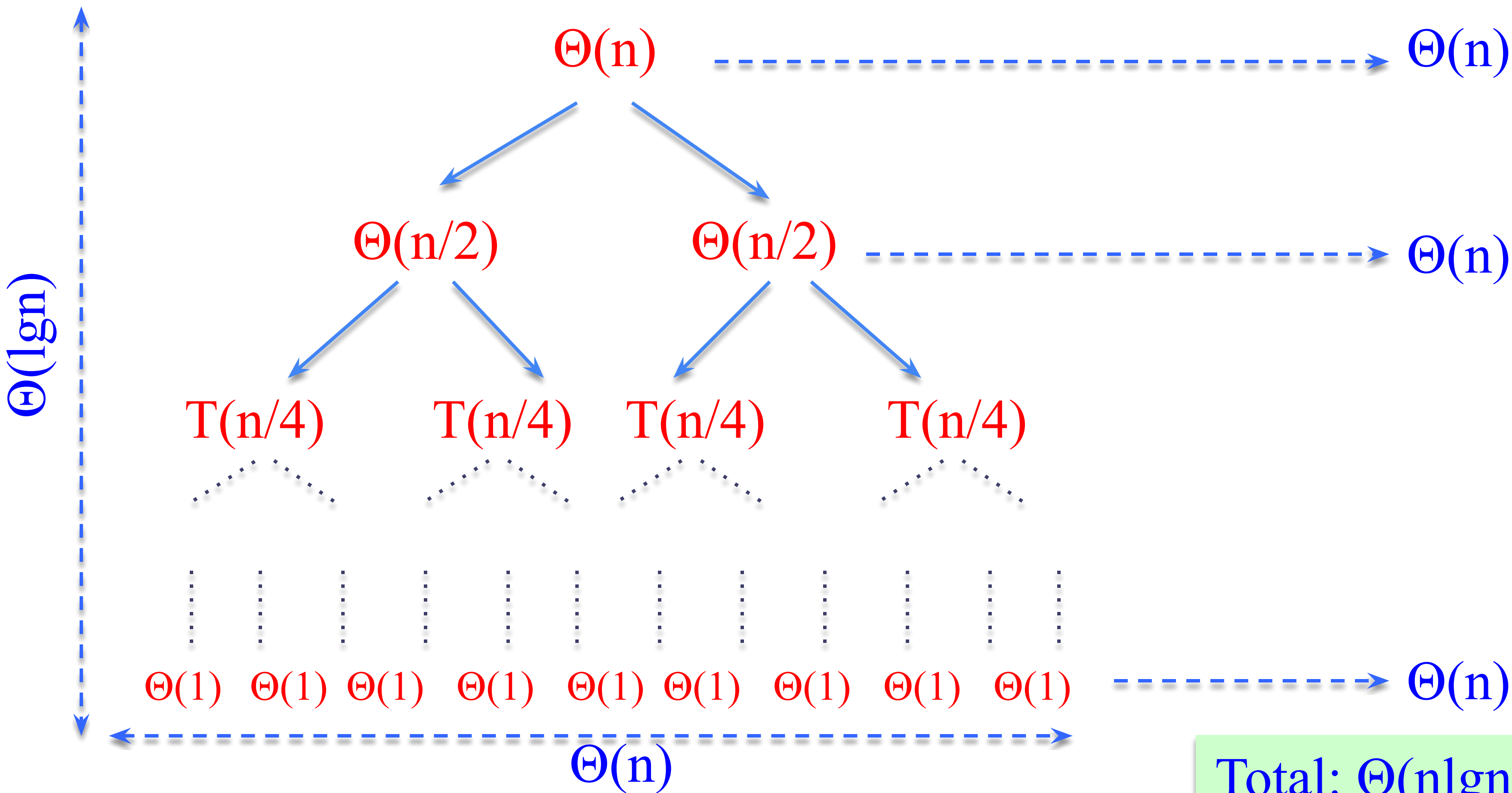
Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Total: $\Theta(n\lg n)$

Merge Sort Complexity

- Recurrence:

  $T(n) = 2T(n/2) + \Theta(n)$

- Solution to recurrence:

  $T(n) = \Theta(n \lg n)$

Conclusions: Insertion Sort vs. Merge Sort

- $\Theta(nlgn)$ grows more slowly than $\Theta(n^2)$
- Therefore Merge-Sort beats Insertion-Sort in the worst case
- In practice, Merge-Sort beats Insertion-Sort for $n>30$ or so.