The Algorithms of Kruskal and Prim
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Both algorithms use a specific rule to:

Determine a safe-edge in the Generic MST algorithm.

In Kruskal’s algorithm, the set $A$ is a forest

The Safe-Edge is always a Least-Weight edge in the graph that connects two distinct components (trees).

In Prim’s algorithm, the set $A$ forms a single tree

The Safe-Edge is always a Least-Weight edge in the graph that connects the tree to a vertex not in tree.
Kruskal’s Algorithm

- Kruskal’s algorithm is based directly on the Generic-MST
- It finds a Safe-Edge to add to the growing forest, by finding an edge \((u,v)\) of Least-Weight of all edges that connect any two trees in the forest
- Let \(C_1\) & \(C_2\) denote two trees that are connected by \((u,v)\)
Kruskal’s Algorithm

• Since \((u,v)\) must be a light-edge connecting \(C_1\) to some other tree, the Corollary implies that \((u,v)\) is a Safe-Edge for \(C_1\).

• Kruskal’s algorithm is a greedy algorithm

  Because at each step it adds to the forest an edge of least possible weight.
The Execution of Kruskal’s Algorithm

(a)
The Execution of Kruskal’s Algorithm

(b)

Diagram of a graph with labeled edges and vertices.
The Execution of Kruskal’s Algorithm

(c)
The Execution of Kruskal’s Algorithm

(d)
The Execution of Kruskal’s Algorithm
The Execution of Kruskal’s Algorithm

(f)
The Execution of Kruskal’s Algorithm

(g)
The Execution of Kruskal’s Algorithm

(h)

The graph shows the execution of Kruskal’s algorithm on a weighted graph. The edge (h,i) is marked as discarded.
The Execution of Kruskal’s Algorithm
The Execution of Kruskal’s Algorithm

(b,c) discarded
The Execution of Kruskal’s Algorithm

(k)

![Graph diagram showing the execution of Kruskal's Algorithm](image)
The Execution of Kruskal’s Algorithm

(e,f) discarded
The Execution of Kruskal’s Algorithm

(b,h) discarded
The Execution of Kruskal’s Algorithm

(d,f) discarded
Kruskal’s Algorithm

• Our implementation of Kruskal’s Algorithm uses a Disjoint-Set Data Structure to maintain several disjoint set of elements

• Each set contains the vertices of a tree of the current forest
Kruskal’s Algorithm

\textbf{MST-KRUSKAL} \,(G, \omega) \\
A \leftarrow \emptyset \\
\text{for each vertex } v \in V[G] \text{ do} \\
\text{MAKE-SET} \,(v) \\
\text{SORT} \text{ the edges of } E \text{ by nondecreasing weight } \omega \\
\text{for each edge } (u,v) \in E \text{ in nondecreasing order do} \\
\text{if } \text{FIND-SET}(u) \neq \text{FIND-SET}(v) \text{ then} \\
A \leftarrow A \cup \{(u,v)\} \\
\text{UNION} \,(u,v) \\
\text{return } A \\
\text{end}
Kruskal’s Algorithm

• The comparison \(\text{FIND-SET}(u) \neq \text{FIND-SET}(v)\) checks whether the endpoints \(u\) & \(v\) belong to the same tree

• If they do, then the edge \((u,v)\) cannot be added to the tree without creating a cycle, and the edge is discarded

• Otherwise, the two vertices belong to different trees, and the edge is added to \(A\)
Running Time of Kruskal’s Algorithm

- The running time for a graph $G = (V, E)$ depends on the implementation of the disjoint-set data structure.
- Use the Disjoint-Set-Forest implementation with the Union-By-Rank and Path-Compression heuristics.
- Since it is the asymptotically fastest implementation known

Initialization (first for-loop) takes time $O(V)$
Sorting takes time $O(E \lg E)$ time
Running Time of Kruskal’s Algorithm

- There are $O(E)$ operations on the disjoint-set forest which in total take $O(E \alpha(E, V))$ time where $\alpha$ is the Functional Inverse of Ackerman’s Function.

- Since $\alpha(E, V) = O(lg E)$

  The total running time is $O(E lg E)$. 
Prim’s Algorithm

- Prim’s algorithm is also a special case of Generic-MST algorithm
- The edges in the set A always form a single tree
- The tree starts from an arbitrary vertex $v$ and grows until the tree spans all the vertices in $V$
- At each step, a light-edge connecting a vertex in $A$ to a vertex in $V$ - $A$ is added to the tree $A$
- Hence, the Corollary implies that Prim’s algorithm adds safe-edges to $A$ at each step.
Prim’s Algorithm

- *This strategy is greedy since*
  The tree is augmented at each step with an edge that contributes the minimum amount possible to the tree’s weight.

- The key to implementing Prim’s algorithm efficiently is to make it easy to select a new edge to be added to A

- All vertices that are not in the tree reside in a priority queue Q based on a key field.

- For each vertex \( v \), \( key[v] \) is the minimum weight of any edge connecting \( v \) to a vertex in the tree
  \( key[v] = \infty \) if there is no such edge.
The Execution of Prim’s Algorithm

(a)
The Execution of Prim’s Algorithm

(b)
The Execution of Prim’s Algorithm

(c)
The Execution of Prim’s Algorithm
The Execution of Prim’s Algorithm

(e)
The Execution of Prim’s Algorithm
The Execution of Prim’s Algorithm

(g)

Diagram showing the execution of Prim's algorithm with vertices and edges labeled.
The Execution of Prim’s Algorithm

(h)
The Execution of Prim’s Algorithm
Prim’s Algorithm

\[ \text{key}[v] = \infty \]
\[ \pi[v] = \text{NIL} \]
Prim’s Algorithm

Vertex $u_1$ moves from $Q$ to $V-Q$ thru $\textsc{extract-min}$
Prim’s Algorithm

Vertex $u_2$ moves from $Q$ to $V-Q$ thru $\text{EXTRACT-MIN}$
Prim’s Algorithm

Vertex $u_3$ moves from $Q$ to $V-Q$ thru $\text{EXTRACT-MIN}$
Prim’s Algorithm

- For each vertex \( v \) we maintain two fields:
  
  \[ \text{key}[v] : \text{Min. weight of any edge connecting } v \text{ to a vertex in the tree.} \]
  
  \[ \text{key}[v] = \infty \text{ if there is no such edge} \]
  
  \[ \pi[v] : \text{Points to the parent of } v \text{ in the tree.} \]

- During the algorithm, the set \( A \) in Generic-MST is maintained as

  \[ A = \{ (v, \pi[v]) : v \in V - \{r\} - Q \} , \text{ where } r \text{ is a random start vertex.} \]

- When the algorithm terminates, the priority queue is empty.

  The \textbf{MST} \( A \) for \( G \) is thus \( A = \{ (v, \pi[v]) : v \in V - \{r\} \} \)
Prim’s Algorithm

**MST-PRIM** \((G, \omega, r)\)

\[ Q \leftarrow V[G] \]

for each \( u \in Q \) do

\[ \text{key}[u] \leftarrow \infty \]

\[ \text{key}[r] \leftarrow 0 \]

\[ \pi[r] \leftarrow \text{NIL} \]

**BUILD-MIN-HEAP** \((Q)\)

while \( Q \neq \emptyset \) do

\[ u \leftarrow \text{EXTRACT-MIN} \ (Q) \]

for each \( v \in \text{Adj} [u] \) do

if \( v \in Q \) and \( \omega(u, v) < \text{key}[v] \) then

\[ \pi[v] \leftarrow u \]

**DECREASE-KEY** \((Q, v, \omega(u, v))\)

/\* \text{key}[v] \leftarrow \omega(u, v) \*/
Prim’s Algorithm

- Through the algorithm, the set $V - Q$ contains the vertices in the tree being grown.

- $u \leftarrow \text{EXTRACT-MIN}(Q)$ identifies a vertex $u \in Q$ incident on a light edge crossing the cut $(V-Q, Q)$ with the exception of the first iteration, in which $u = r$

- Removing $u$ from the set $Q$ adds it to the set $V - Q$ of vertices in the tree
Prim’s Algorithm

- The inner for-loop updates the key & π fields of every vertex v adjacent to u but not in the tree

- This updating maintains the invariants

\[
\text{key}[v] \leftarrow \omega(v, \pi[v]), \text{ and}
\]

\[
(\nu, \pi[v]) \text{ is a light-edge connecting } \nu \text{ to the tree}
\]
Running Time of Prim’s Algorithm

• The performance of Prim’s algorithm depends on how we implement the priority queue

• If Q is implemented as a binary heap

  Use **BUILD-MIN-HEAP** procedure to perform the initialization in $O(V)$ time

  **while-loop** is executed $|V|$ times

  each **EXTRACT-MIN** operation takes $O(lgV)$ time

Therefore, the total time for all calls **EXTRACT-MIN** is $O(V lg V)$
Running Time of Prim’s Algorithm

• The inner for-loop is executed O(E) times altogether since the sum of the lengths of all adjacency lists is 2|E|

• Within the for-loop

  The membership test $v \in Q$ can be implemented in constant time by keeping a bit for each vertex whether or not it is in Q and updating the bit when vertex is removed from Q

  The assignment $\text{key}[v] \leftarrow \omega(u, v)$ involves a DECREASE-KEY operation on the heap which can be implemented in $O(\lg V)$ time
Running Time of Prim’s Algorithm

• Thus, the total time for Prim’s algorithm is

\[ O( V \lg V + E \lg V ) = O( E \lg V ) \]

• The asymptotic running time of Prim’s algorithm can be improved by using **FIBONACCI HEAPS**

• If \(|V|\) elements are organized into a fibonacci heap we can perform:

  An **EXTRACT-MIN** operation in \(O(\lg V)\) amortized time
  A **DECREASE-KEY** operation (line 11) in \(O(1)\) amortized time
Running Time of Prim’s Algorithm

The asymptotic running time of Prim’s algorithm can be improved by using **FIBONACCI HEAPS**

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- An **EXTRACT-MIN** operation in $O(\lg V)$ amortized time
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Hence, if we use **FIBONACCI-HEAP** to implement the priority queue $Q$ the running time of Prim’s algorithm improves to:

$$O( E + V \lg V )$$