# CS473 - Algorithms I

## Lecture 4
## The Divide-and-Conquer Design Paradigm

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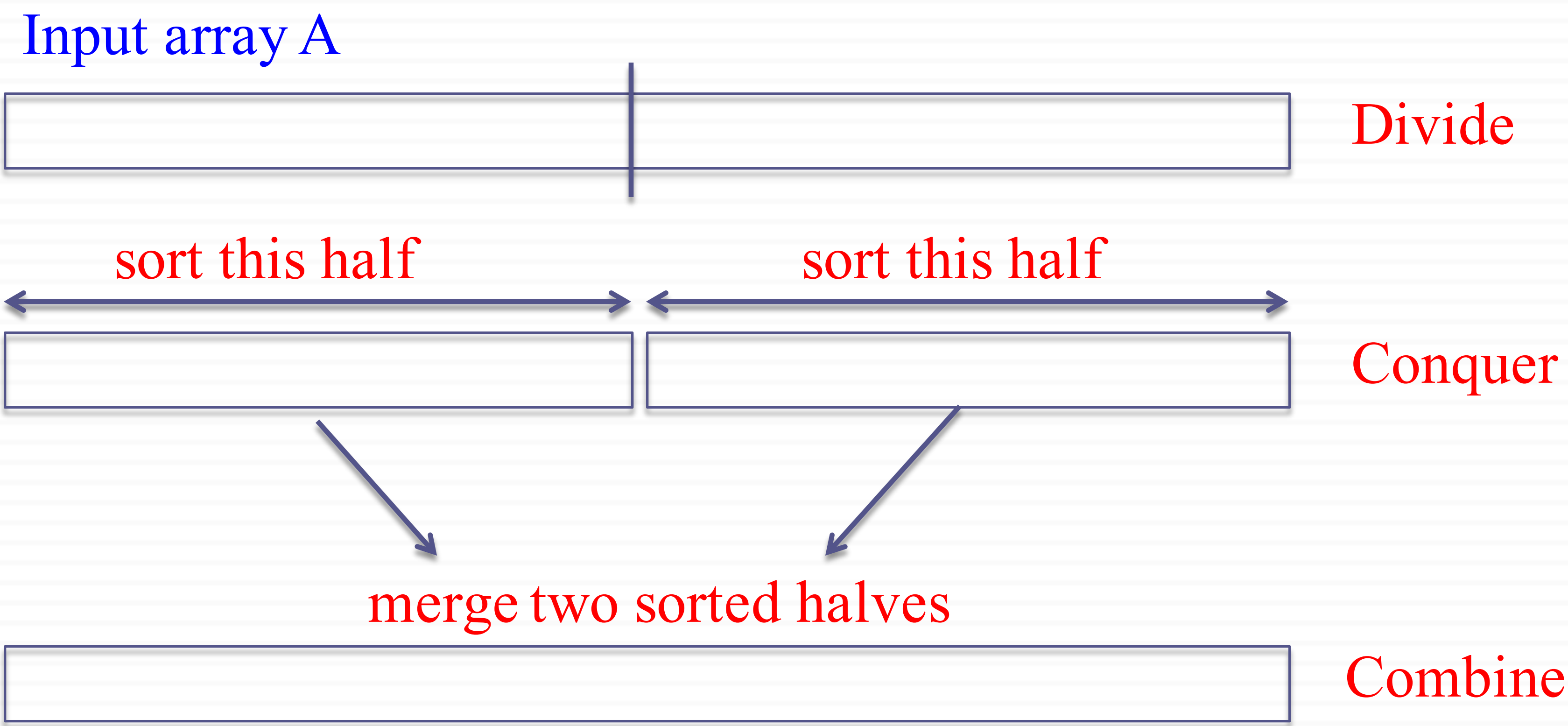
# Reminder: Merge Sort



# The Divide-and-Conquer Design Paradigm

1. ***Divide*** **the problem (instance) into subproblems.**
2. ***Conquer*** **the subproblems by solving them recursively.**
3. ***Combine*** **subproblem solutions.**

# Example: Merge Sort

1. ***Divide***: Trivial.
2. ***Conquer***: Recursively sort 2 subarrays.
3. ***Combine***: Linear- time merge.

$$T(n) = \underbrace{2}_{\text{\# subproblems}}\, T(\underbrace{n/2}_{\text{subproblem size}}) + \underbrace{\Theta(n)}_{\text{work dividing and combining}}$$

# Master Theorem: Reminder

## T(n) = aT(n/b) + f(n)

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ → $T(n) = \Theta(n^{\log_b a})$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ → $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Case 3: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$

and $a f(n/b) \leq c f(n)$ for $c < 1$ → $T(n) = \Theta(f(n))$

# Merge Sort: Solving the Recurrence

$$T(n) = 2\,T(n/2) + \Theta(n)$$

$\Rightarrow$ $a = 2, \quad b = 2, \quad f(n) = \Theta(n), \quad n^{\log_b a} = n$

<u>Case 2</u>: $\boxed{\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)} \Rightarrow \boxed{T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)}$

*holds for k = 0*

$\Rightarrow$ $T(n) = \Theta\,(n\lg n)$

# Binary Search

Find an element in a sorted array:

*1.* ***Divide***: Check middle element.

*2.* ***Conquer***: Recursively search 1 subarray.

*3.* ***Combine***: Trivial.

***Example:*** Find **9**

| | | | | | | |
|---|---|---|---|---|---|---|
| **3** | **5** | **7** | **8** | **9** | **12** | **15** |

# Binary Search

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# Binary Search

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# Binary Search

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*1.* ***Divide*:** Check middle element.

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***Example:*** Find **9**

| | | | | | | |
|---|---|---|---|---|---|---|
| **3** | **5** | **7** | **8** | **9** | **12** | **15** |

# Recurrence for Binary Search

$$T(n) = 1\, T(n/2) + \Theta(1)$$

**# subproblems**

**subproblem size**

**work dividing and combining**

# Binary Search: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

$\Rightarrow$ $a = 1$, $b = 2$, $f(n) = \Theta(1)$, $n^{\log_b a} = n^0 = 1$

Case 2: $\frac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

*holds for $k = 0$*

$\Rightarrow$ $T(n) = \Theta(\lg n)$

# Powering a Number

- Problem: Compute $a^n$, where $n$ is a natural number

```
Naive-Power (a, n)
    powerVal ← 1
    for i ← 1 to n
        powerVal ← powerVal . a
    return powerVal
```

- What is the complexity?

$T(n) = \Theta(n)$


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# Powering a Number: Divide & Conquer

Basic idea:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is } \underline{\text{even}} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is } \underline{\text{odd}} \end{cases}$$

# Powering a Number: Divide & Conquer

```
POWER (a, n)
        if n = 0 then return 1

        else if n is even then
                val ← POWER (a, n/2)
                return val * val

        else if n is odd then
                val ← POWER (a, (n-1)/2)
                return val * val * a
```

# Powering a Number: Solving the Recurrence

$$T(n) = T(n/2) + \Theta(1)$$

$\Rightarrow$ $a = 1$, $b = 2$, $f(n) = \Theta(1)$, $n^{\log_b a} = n^0 = 1$

Case 2: $\dfrac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

*holds for k = 0*

$\Rightarrow$ $T(n) = \Theta(\lg n)$

# Matrix Multiplication

**Input :** $A = [a_{ij}]$, $B = [b_{ij}]$.
**Output:** $C = [c_{ij}] = A \cdot B$.

$i, j = 1, 2, \ldots, n.$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$c_{ij} = \sum_{1 \le k \le n} a_{ik} . b_{kj}$$

# Standard Algorithm

for $i \leftarrow 1$ to $n$

do for $j \leftarrow 1$ to $n$

do $c_{ij} \leftarrow 0$

for $k \leftarrow 1$ to $n$

do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time = $\Theta(n^3)$

# Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} & = & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \cdot & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{array}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21}$$

# Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right) \end{array}$$

$$c_{12} = a_{11} b_{12} + a_{12} b_{22}$$

# Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\overset{C}{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}} = \overset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \cdot \overset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}$$

$$c_{21} = a_{21} b_{11} + a_{22} b_{21}$$

# Matrix Multiplication: Divide & Conquer

IDEA: Divide the n x n matrix into

2x2 matrix of (n/2)x(n/2) submatrices

$$\begin{array}{ccccc} C & & A & & B \\ \left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & \boxed{c_{22}} \end{array}\right) & = & \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline \boxed{a_{21}} & \boxed{a_{22}} \end{array}\right) & \cdot & \left(\begin{array}{c|c} b_{11} & \boxed{b_{12}} \\ \hline b_{21} & \boxed{b_{22}} \end{array}\right) \end{array}$$

$$c_{22} = a_{21}\, b_{12} + a_{22}\, b_{22}$$

# Matrix Multiplication: Divide & Conquer

$$\overset{C}{\left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right)} = \overset{A}{\left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right)} \cdot \overset{B}{\left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right)}$$

$$\left.\begin{aligned} c_{11} &= a_{11} b_{11} + a_{12} b_{21} \\ c_{12} &= a_{11} b_{12} + a_{12} b_{22} \\ c_{21} &= a_{21} b_{11} + a_{22} b_{21} \\ c_{22} &= a_{21} b_{12} + a_{22} b_{22} \end{aligned}\right\}$$

8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices

# Matrix Multiplication: Divide & Conquer

MATRIX-MULTIPLY (A, B)

*// Assuming that both A and B are nxn matrices*

**if** n = 1 **then return** A * B

**else**

partition A, B, and C as shown before

$c_{11}$ = MATRIX-MULTIPLY($a_{11}$, $b_{11}$) + MATRIX-MULTIPLY($a_{12}$, $b_{21}$)

$c_{12}$ = MATRIX-MULTIPLY($a_{11}$, $b_{12}$) + MATRIX-MULTIPLY($a_{12}$, $b_{22}$)

$c_{21}$ = MATRIX-MULTIPLY($a_{21}$, $b_{11}$) + MATRIX-MULTIPLY($a_{22}$, $b_{21}$)

$c_{22}$ = MATRIX-MULTIPLY($a_{21}$, $b_{12}$) + MATRIX-MULTIPLY($a_{22}$, $b_{22}$)

**return** C

# Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

8 recursive calls

each subproblem has size n/2

submatrix addition

# Matrix Multiplication: Solving the Recurrence

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

$\Rightarrow$ $a = 8$, $b = 2$, $f(n) = \Theta(n^2)$, $n^{\log_b a} = n^3$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$\Rightarrow$ $T(n) = \Theta\,(n^3)$

*No better than the ordinary algorithm!*

# Matrix Multiplication: Strassen's Idea

$$\overset{C}{\left(\begin{array}{c|c} c_{11} & c_{12} \\ \hline c_{21} & c_{22} \end{array}\right)} = \overset{A}{\left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right)} \cdot \overset{B}{\left(\begin{array}{c|c} b_{11} & b_{12} \\ \hline b_{21} & b_{22} \end{array}\right)}$$

Compute $\mathbf{c_{11}}$, $\mathbf{c_{12}}$, $\mathbf{c_{21}}$, and $\mathbf{c_{22}}$ using **7** recursive multiplications

# Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$

$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$

$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$

$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$

$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$

$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$

$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$

Reminder: Each submatrix is of size $(n/2)x(n/2)$

Each add/sub operation takes $\Theta(n^2)$ time

Compute $P_1..P_7$ using 7 recursive calls to matrix-multiply

*How to compute $c_{ij}$ using $P_1.. P_7$ ?*

# Matrix Multiplication: Strassen's Idea

$$P_1 = a_{11} \mathbf{x} (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) \mathbf{x} b_{22}$$

$$P_3 = (a_{21} + a_{22}) \mathbf{x} b_{11}$$

$$P_4 = a_{22} \mathbf{x} (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) \mathbf{x} (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) \mathbf{x} (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) \mathbf{x} (b_{11} + b_{12})$$

$$c_{11} = P_5 + P_4 - P_2 + P_6$$

$$c_{12} = P_1 + P_2$$

$$c_{21} = P_3 + P_4$$

$$c_{22} = P_5 + P_1 - P_3 - P_7$$

7 recursive multiply calls

18 add/sub operations

*Does not rely on commutativity of multiplication*



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# Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} \times (b_{12} - b_{22})$

$P_2 = (a_{11} + a_{12}) \times b_{22}$

$P_3 = (a_{21} + a_{22}) \times b_{11}$

$P_4 = a_{22} \times (b_{21} - b_{11})$

$P_5 = (a_{11} + a_{22}) \times (b_{11} + b_{22})$

$P_6 = (a_{12} - a_{22}) \times (b_{21} + b_{22})$

$P_7 = (a_{11} - a_{21}) \times (b_{11} + b_{12})$

e.g. Show that $c_{12} = P_1 + P_2$

$$
\begin{aligned}
c_{12} &= P_1 + P_2 \\
&= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22} \\
&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\
&= a_{11}b_{12} + a_{12}b_{22}
\end{aligned}
$$

# Strassen's Algorithm

1. **Divide**: Partition A and B into $(n/2) \times (n/2)$ submatrices. Form terms to be multiplied using + and –.

2. **Conquer**: Perform **7** multiplications of $(n/2) \times (n/2)$ submatrices recursively.

3. **Combine**: Form C using + and – on $(n/2) \times (n/2)$ submatrices.

Recurrence: $T(n) = 7\,T(n/2) + \Theta(n^2)$

# Strassen's Algorithm: Solving the Recurrence

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$\Rightarrow$ $a = 7$, $b = 2$, $f(n) = \Theta(n^2)$, $n^{\log_b a} = n^{\lg 7}$

Case 1: $\dfrac{n^{\log_b a}}{f(n)} = \Omega(n^{\varepsilon})$ $\Rightarrow$ $T(n) = \Theta(n^{\log_b a})$

$\Rightarrow$ $T(n) = \Theta\,(n^{\lg 7})$

*Note*: $\lg 7 \approx 2.81$

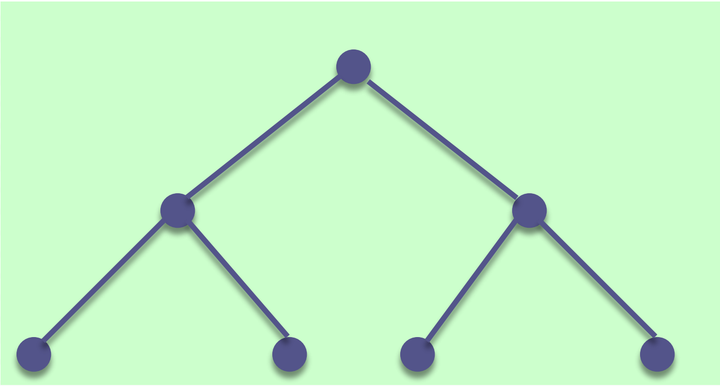
# Strassen's Algorithm

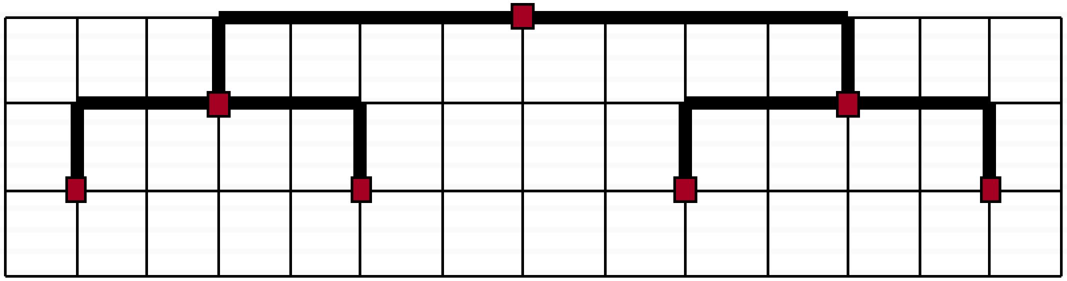
- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376...})$ *(of theoretical interest only)*

# VLSI Layout: Binary Tree Embedding

- Problem: Embed a complete binary tree with n leaves into a 2D grid with minimum area.

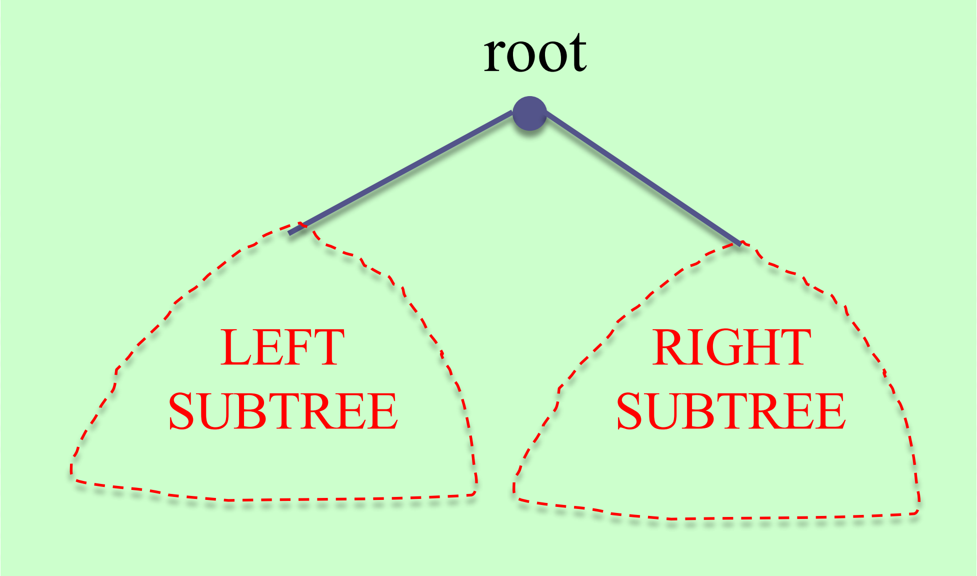
- Example:





# Binary Tree Embedding

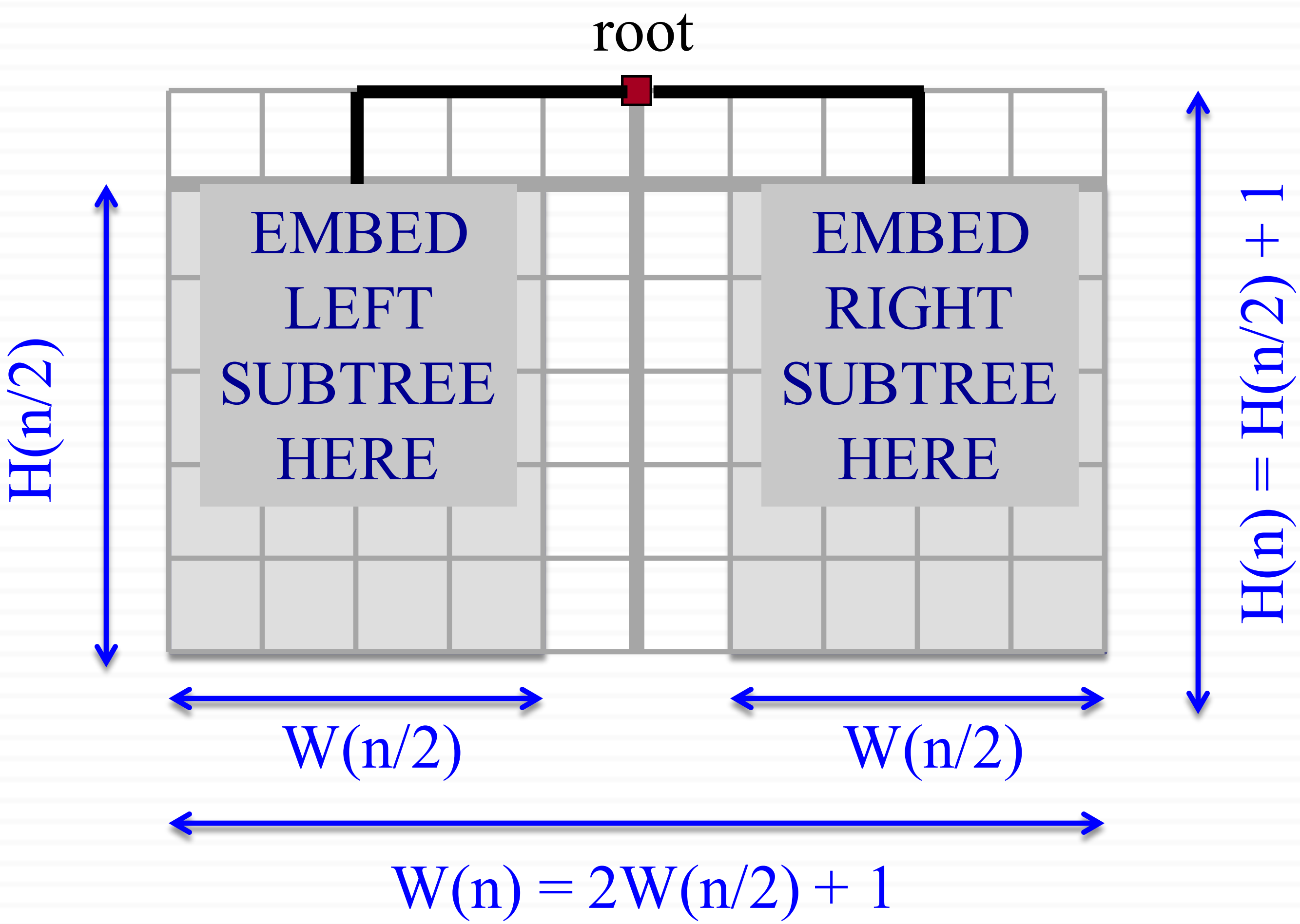
- Use divide and conquer




1. Embed the root node
2. Embed the left subtree
3. Embed the right subtree

What is the min-area required for n leaves?

# Binary Tree Embedding



# Binary Tree Embedding

- Solve the recurrences:

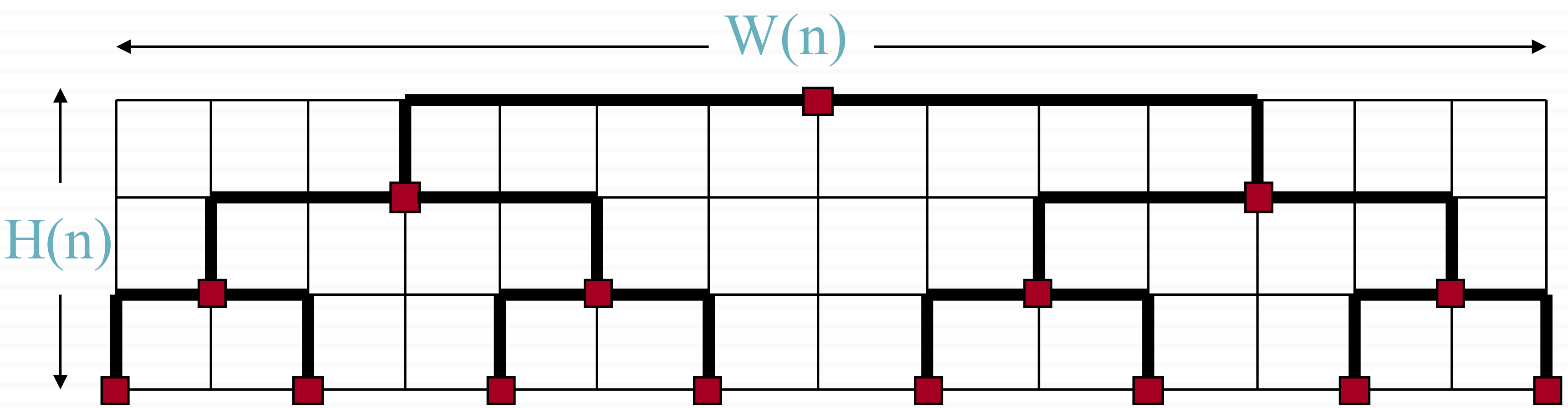
  $$W(n) = 2W(n/2) + 1$$
  $$H(n) = H(n/2) + 1$$

  $$\rightarrow W(n) = \Theta(n)$$
  $$\rightarrow H(n) = \Theta(\lg n)$$

- $Area(n) = \Theta(n\lg n)$

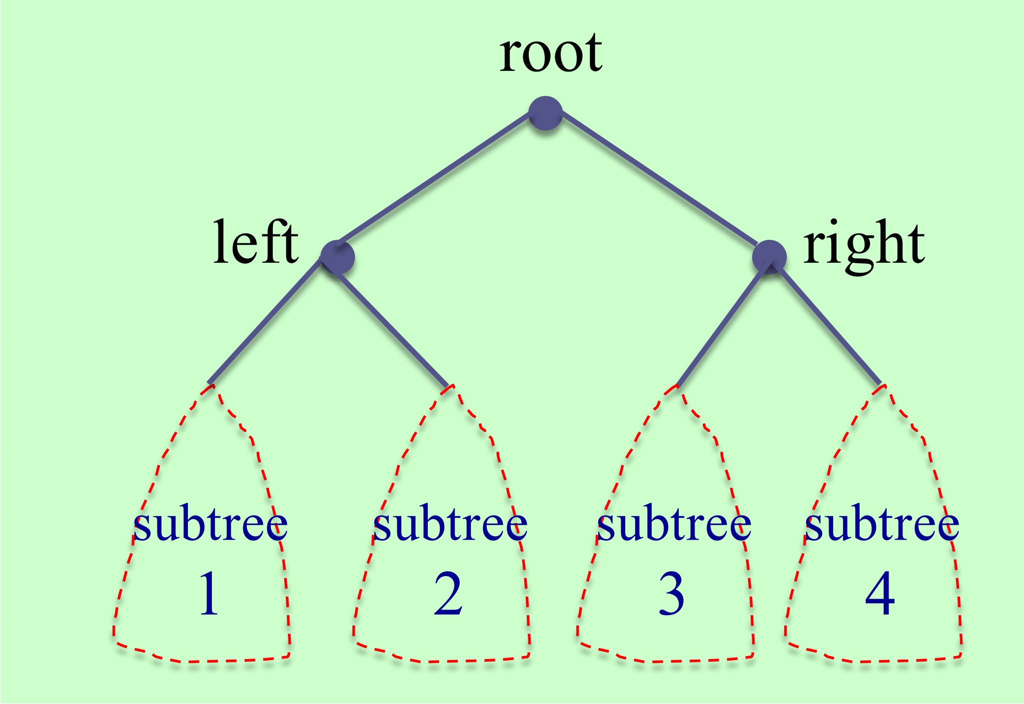
# Binary Tree Embedding

Example:



# Binary Tree Embedding: H-Tree

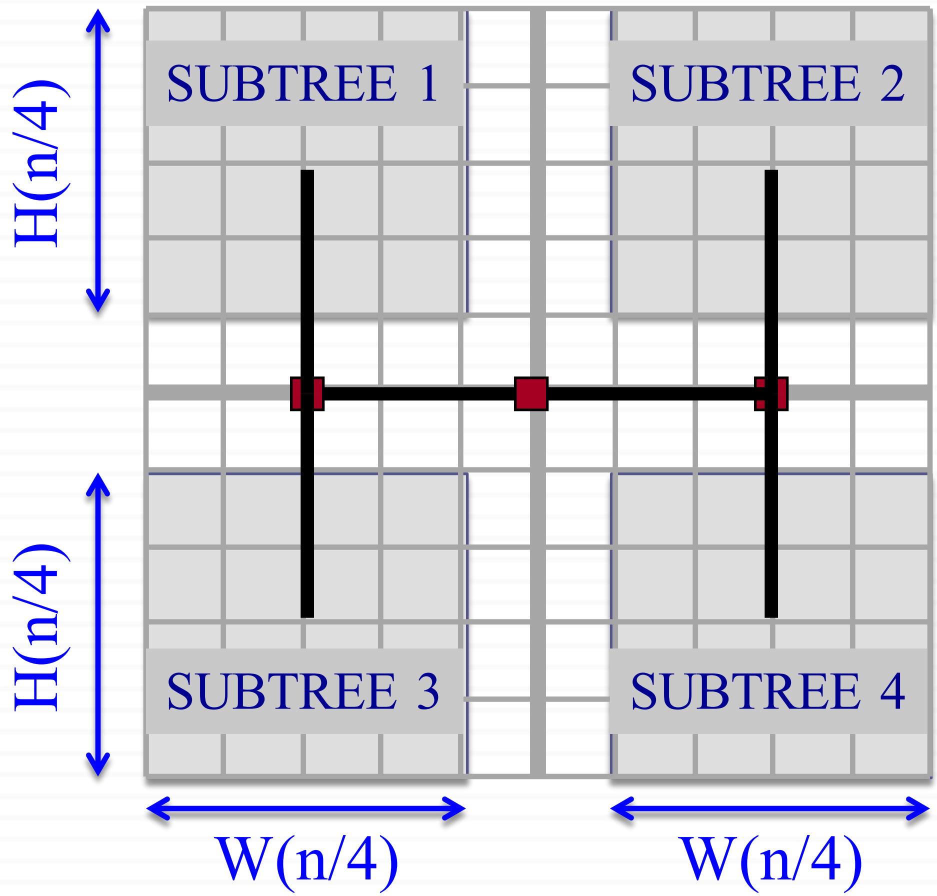
- Use a different divide and conquer method




1. Embed root, left, right nodes
2. Embed subtree 1
3. Embed subtree 2
4. Embed subtree 3
5. Embed subtree 4

*What is the min-area required for n leaves?*

# Binary Tree Embedding: H-Tree




$$W(n) = 2W(n/4) + 1$$

$$H(n) = 2H(n/4) + 1$$

# Binary Tree Embedding: H-Tree

- Solve the recurrences:

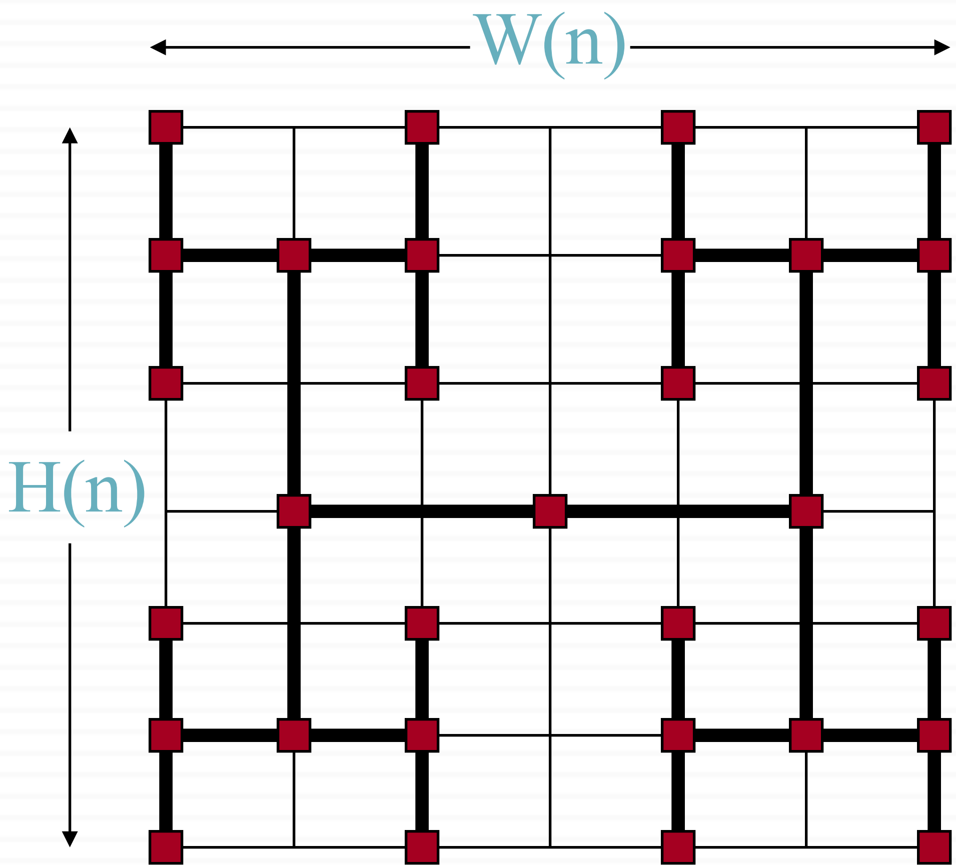
  $$W(n) = 2W(n/4) + 1$$
  $$H(n) = 2H(n/4) + 1$$

  $$\Rightarrow W(n) = \Theta(\sqrt{n})$$
  $$\Rightarrow H(n) = \Theta(\sqrt{n})$$

- $\text{Area}(n) = \Theta(n)$

# Binary Tree Embedding: H-Tree

Example:



# Correctness Proofs

- ***Proof by induction*** commonly used for D&C algorithms

- *Base case*: Show that the algorithm is correct when the recursion bottoms out (i.e., for sufficiently small n)

- *Inductive hypothesis*: Assume the alg. is correct for any recursive call on any smaller subproblem of size $k$ ($k < n$)

- *General case*: Based on the inductive hypothesis, prove that the alg. is correct for any input of size $n$

# Example Correctness Proof: Powering a Number

```
POWER (a, n)
        if n = 0 then return 1

        else if n is even then
                val ← POWER (a, n/2)
                return val * val

        else if n is odd then
                val ← POWER (a, (n-1)/2)
                return val * val * a
```

# Example Correctness Proof: Powering a Number

- *Base case*: POWER (a, 0) is correct, because it returns 1
- *Ind. hyp*: Assume POWER (a, k) is correct for any $k < n$
- *General case*:

In POWER (a, n) function:

If n is even:

$val = a^{n/2}$ *(due to ind. hyp.)*

it returns $val \,.\, val = a^n$

If n is odd:

$val = a^{(n-1)/2}$ *(due to ind. hyp.)*

it returns $val.\, val \,.\, a = a^n$

➔ *The correctness proof is complete*

# Maximum Subarray Problem

- *Input*: An array of values
- *Output*: The contiguous subarray that has the largest sum of elements

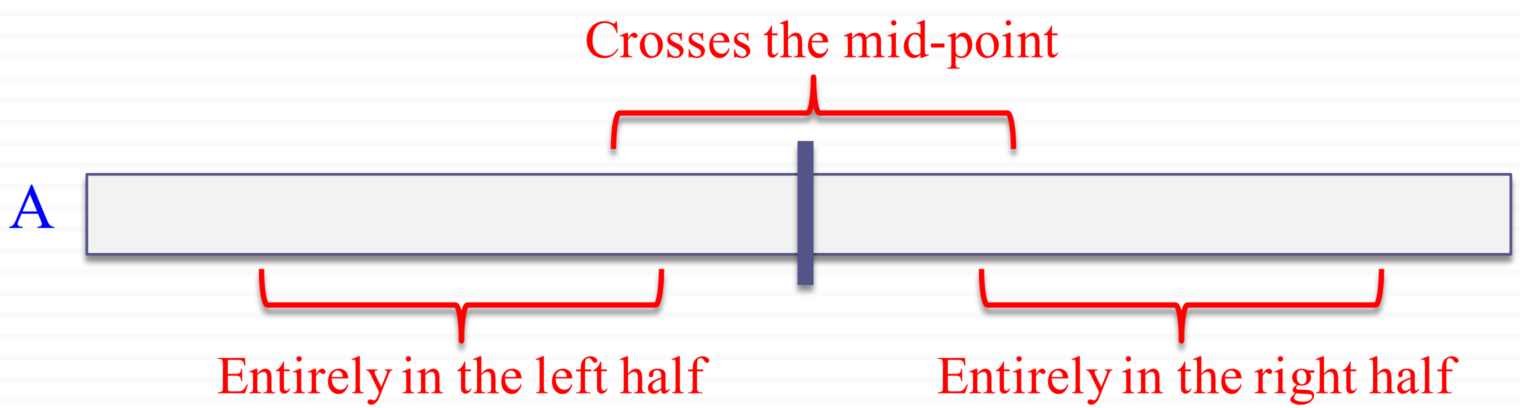
Input array:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | -3 | -25 | 20 | -3 | -16 | -23 | 18 | 20 | -7 | 12 | -22 | -4 | 7 |

the maximum contiguous subarray

# Maximum Subarray Problem: Divide & Conquer

- ***Basic idea***:
  - Divide the input array into 2 from the middle
  - Pick the best solution among the following:
    1. The max subarray of the left half
    2. The max subarray of the right half
    3. The max subarray crossing the mid-point



# Maximum Subarray Problem: Divide & Conquer

- *Divide*: Trivial (divide the array from the middle)
- *Conquer*: Recursively compute the max subarrays of the left and right halves
- *Combine*: Compute the max-subarray crossing the mid-point *(can be done in $\Theta(n)$ time)*. Return the max among the following:
  1. the max subarray of the left subarray
  2. the max subarray of the right subarray
  3. the max subarray crossing the mid-point

See textbook for the detailed solution.

# Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms