CS473 - Algorithms I

Lecture 1

Introduction to Analysis of Algorithms

View in slide-show mode

Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Algorithm Definition

- Algorithm: A sequence of computational steps that transform the input to the desired output
- Procedure vs. algorithm
  - *An algorithm **must halt within finite time** with the right output*
- Example:

a sequence of n numbers → Sorting Algorithm → **sorted permutation of input sequence**

Many Real World Applications

- Bioinformatics
  - Determine/compare DNA sequences
- Internet
  - Manage/manipulate/route data
- Information retrieval
  - Search and access information in large data
- Security
  - Encode & decode personal/financial/confidential data
- Electronic design automation
  - Minimize human effort in chip-design process

Course Objectives

- Learn basic algorithms & data structures
- Gain skills to design new algorithms

- Focus on efficient algorithms

- Design algorithms that
  - are fast
  - use as little memory as possible
  - are correct!

Outline of Lecture 1

- Study two sorting algorithms as examples
  - Insertion sort: *Incremental* algorithm
  - Merge sort: *Divide-and-conquer*

- Introduction to runtime analysis
  - Best vs. worst vs. average case
  - Asymptotic analysis

Sorting Problem

Input: Sequence of numbers

$$\langle a_1, a_2, \ldots, a_n \rangle$$

Output: A permutation

$$\Pi = \langle \Pi(1), \Pi(2), \ldots, \Pi(n) \rangle$$

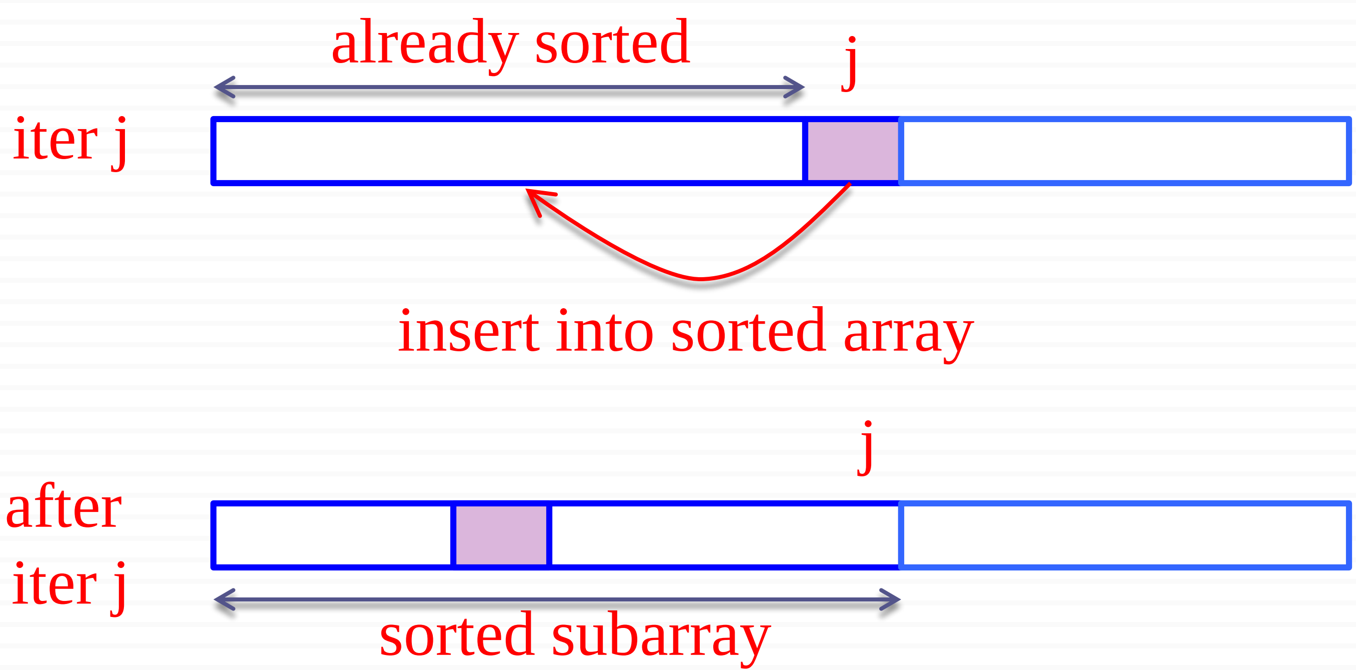
such that

$$a_{\Pi(1)} \leq a_{\Pi(2)} \leq \ldots \leq a_{\Pi(n)}$$

Insertion Sort

Insertion Sort: Basic Idea

- Assume input array: A[1..n]
- Iterate j from 2 to n



Pseudo-code notation

- Objective: Express algorithms to humans in a clear and concise way

- Liberal use of English

- Indentation for block structures

- Omission of error handling and other details

  → *needed in real programs*

Algorithm: Insertion Sort (from Section 2.2)

Insertion-Sort (A)

1. **for** j ← 2 **to** n **do**
2. key ← A[j];
3. i ← j - 1;
4. **while** i > 0 **and** A[i] > key
   do
5. A[i+1] ← A[i];
6. i ← i - 1;
   endwhile
7. A[i+1] ← key;

endfor

Algorithm: Insertion Sort

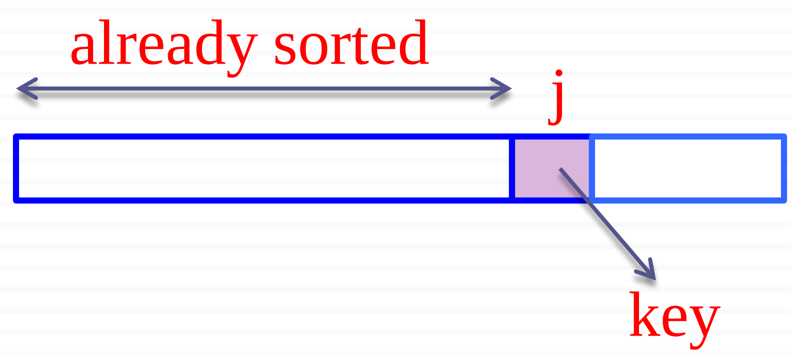
Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```

Iterate over array elts j

Loop invariant:

The subarray A[1..j-1] is always sorted

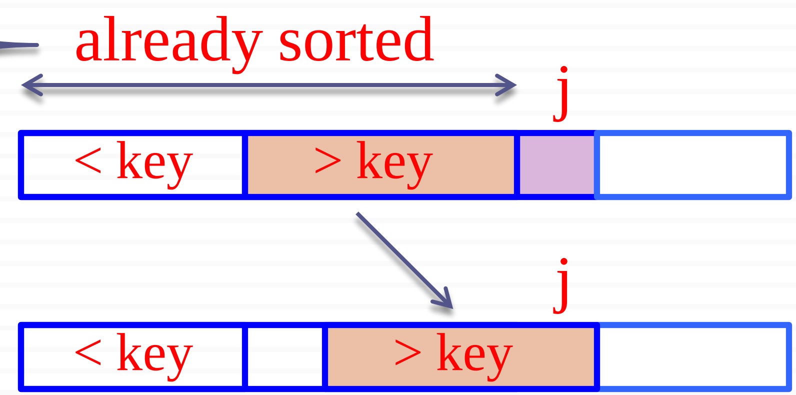


Algorithm: Insertion Sort

<u>Insertion-Sort</u> (A)

```
1. for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
   endfor
```

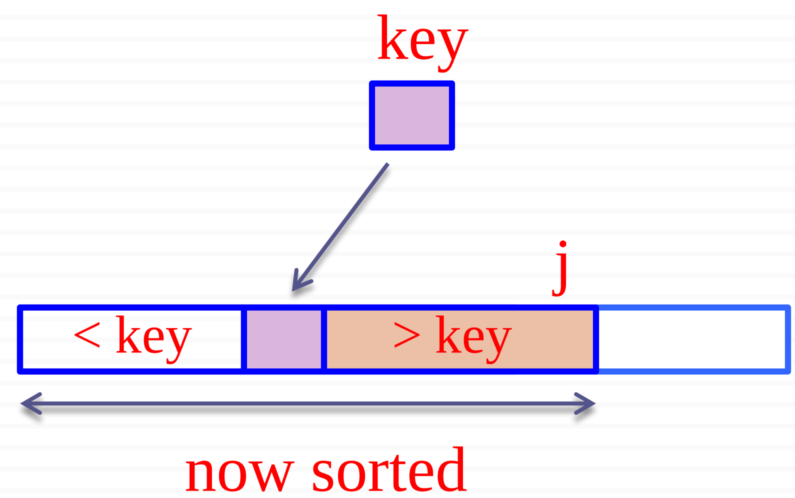
Shift right the entries
in A[1..j-1] that are > key



Algorithm: Insertion Sort

Insertion-Sort (A)

```
1. for j ← 2 to n do
2.       key ← A[j];
3.       i ← j - 1;
4.       while i > 0 and A[i] > key
         do
5.           A[i+1] ← A[i];
6.           i ← i - 1;
         endwhile
7.       A[i+1] ← key;
   endfor
```



Insert key to the correct location

End of iter j: A[1..j] is sorted

Insertion Sort - Example

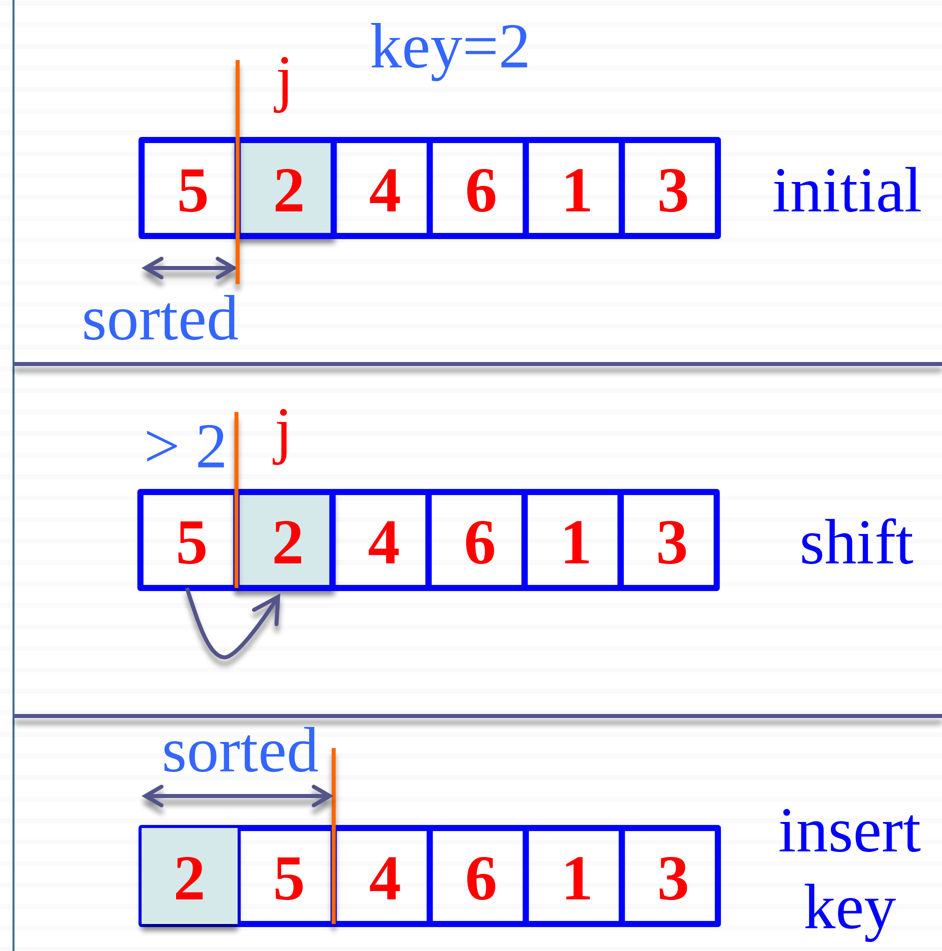
```
Insertion-Sort (A)
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```

5	**2**	**4**	**6**	**1**	**3**

Insertion Sort - Example: Iteration j=2

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
          do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



Insertion Sort - Example: Iteration j=3

Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
   do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```

key=4

j

2	**5**	**4**	**6**	**1**	**3**	initial

sorted

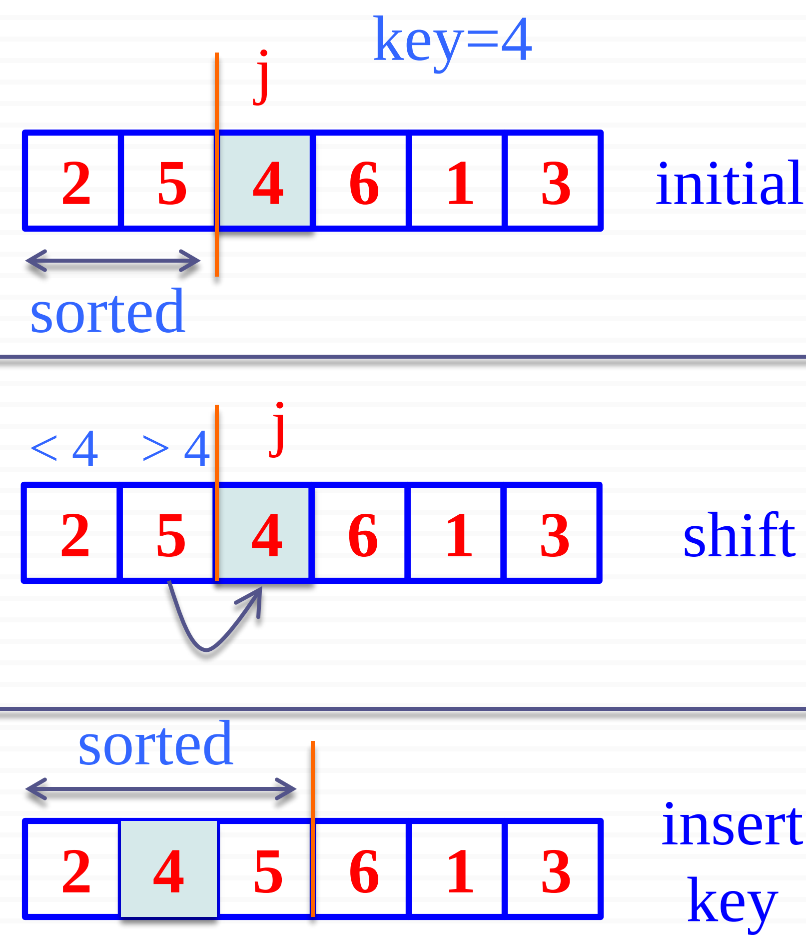
What are the entries at the end of iteration j=3?

?	**?**	**?**	**?**	**?**	**?**

Insertion Sort - Example: Iteration j=3

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```

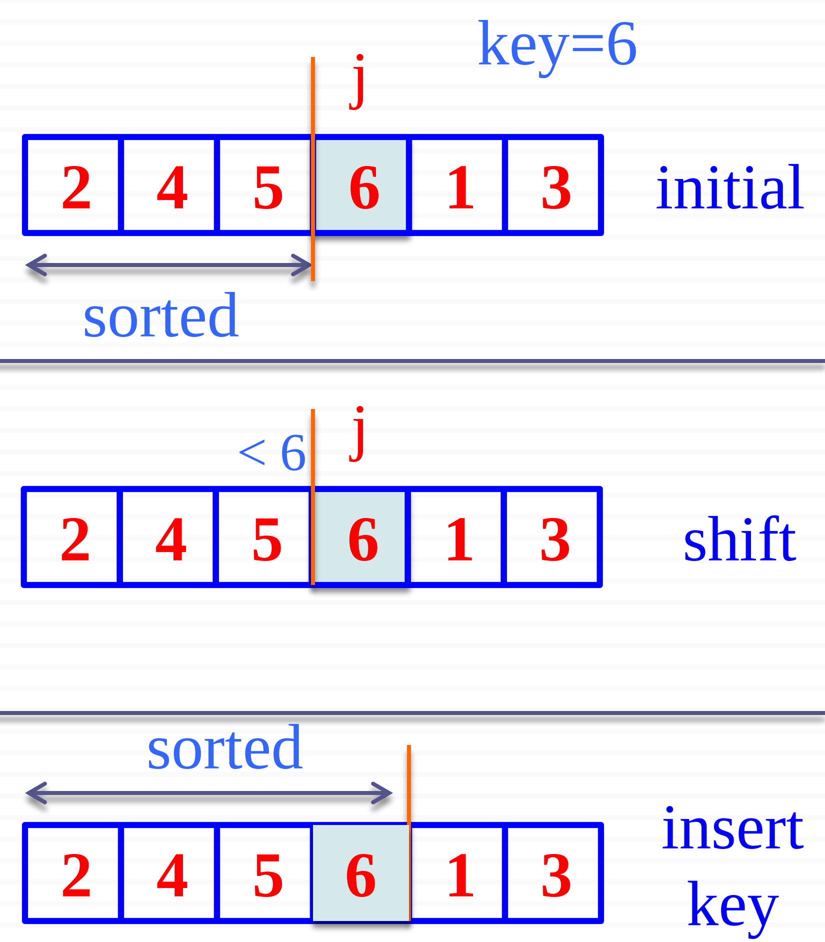


Cevdet Aykanat and Mustafa Ozdal
Computer Engineering Department, Bilkent University

Insertion Sort - Example: Iteration j=4

Insertion-Sort (A)

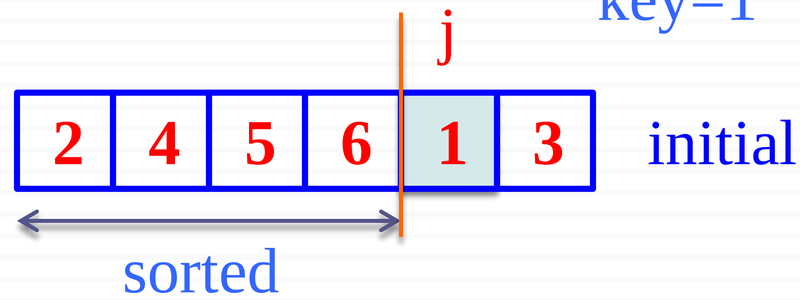
```
1.  for j ← 2 to n do
2.        key ← A[j];
3.        i ← j - 1;
4.        while i > 0 and A[i] > key
          do
5.              A[i+1] ← A[i];
6.              i ← i - 1;
          endwhile
7.        A[i+1] ← key;
    endfor
```



Insertion Sort - Example: Iteration j=5

```
Insertion-Sort (A)
1.  for j ← 2 to n do
2.       key ← A[j];
3.       i ← j - 1;
4.       while i > 0 and A[i] > key
         do
5.            A[i+1] ← A[i];
6.            i ← i - 1;
         endwhile
7.       A[i+1] ← key;
    endfor
```

key=1



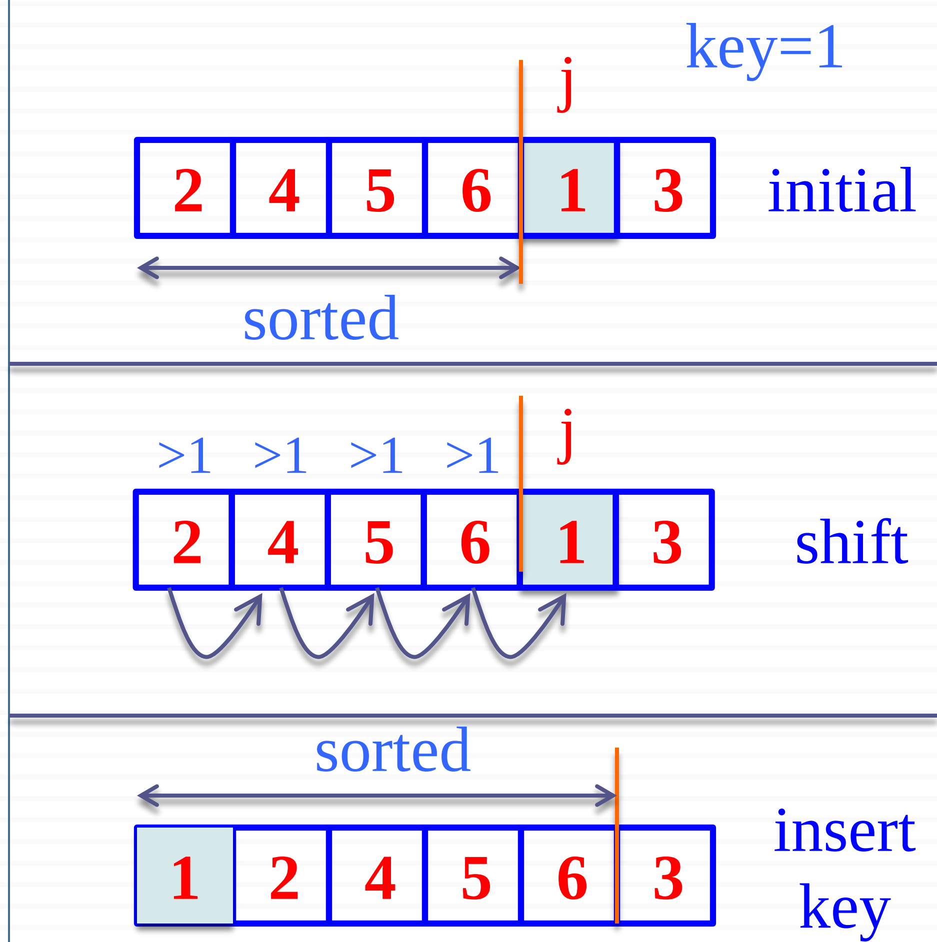
What are the entries at the end of iteration j=5?

?	?	?	?	?	?

Insertion Sort - Example: Iteration j=5

<u>Insertion-Sort</u> (A)

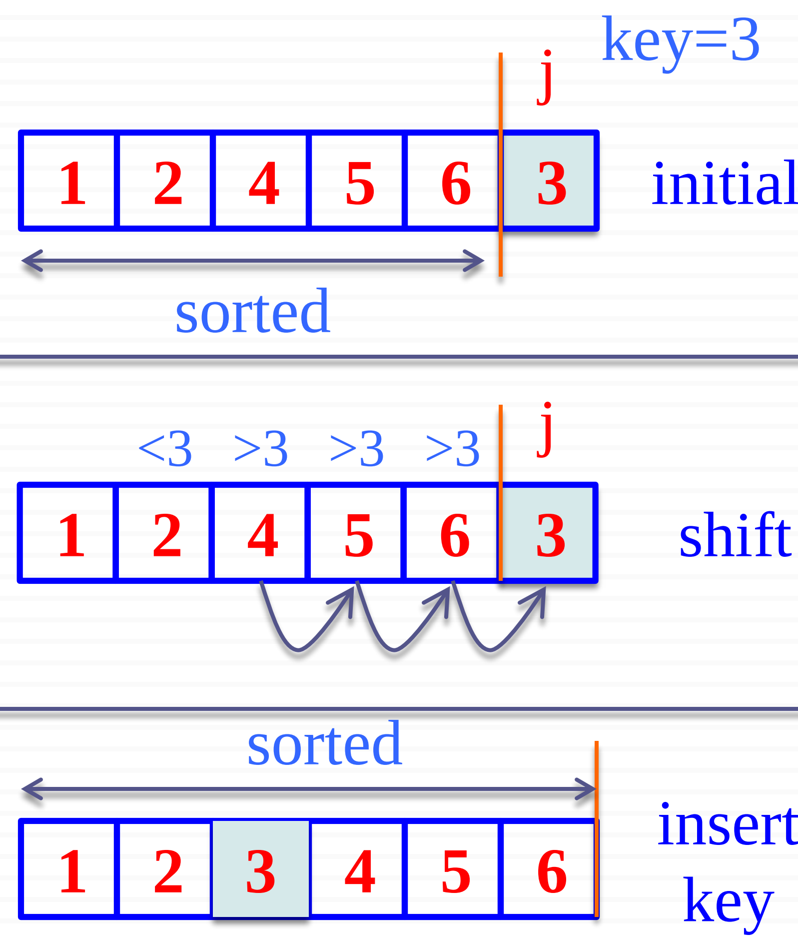
```
1. for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
        do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
   endfor
```



Insertion Sort - Example: Iteration j=6

Insertion-Sort (A)

```
1.  for j ← 2 to n do
2.      key ← A[j];
3.      i ← j - 1;
4.      while i > 0 and A[i] > key
          do
5.          A[i+1] ← A[i];
6.          i ← i - 1;
        endwhile
7.      A[i+1] ← key;
    endfor
```



Insertion Sort Algorithm - Notes

- Items sorted in-place
  - Elements rearranged within array
  - At most constant number of items stored outside the array at any time (e.g. the variable *key*)
  - Input array A contains sorted output sequence when the algorithm ends

- Incremental approach
  - Having sorted A[1..j-1], place A[j] correctly so that A[1..j] is sorted

Running Time

- Depends on:
  - Input size (e.g., 6 elements vs 6M elements)
  - Input itself (e.g., partially sorted)

- Usually want *upper bound*

Kinds of running time analysis

- Worst Case *(Usually)*

  $T(n)$ = max time on any input of size n

- Average Case (*Sometimes*)

  $T(n)$ = average time over all inputs of size n

  Assumes statistical distribution of inputs

- Best Case (*Rarely*)

  $T(n)$ = min time on any input of size n

  BAD*: Cheat with slow algorithm that works fast on some inputs

  GOOD: Only for showing bad lower bound

*Can modify any algorithm (almost) to have a low best-case running time

- Check whether input constitutes an output at the very beginning of the algorithm

Running Time

- For Insertion-Sort, what is its worst-case time?
  - Depends on speed of primitive operations
    - Relative speed (on same machine)
    - Absolute speed (on different machines)

- Asymptotic analysis
  - Ignore machine-dependent constants
  - Look at growth of $T(n)$ as $n \to \infty$

Θ Notation

- Drop low order terms
- Ignore leading constants

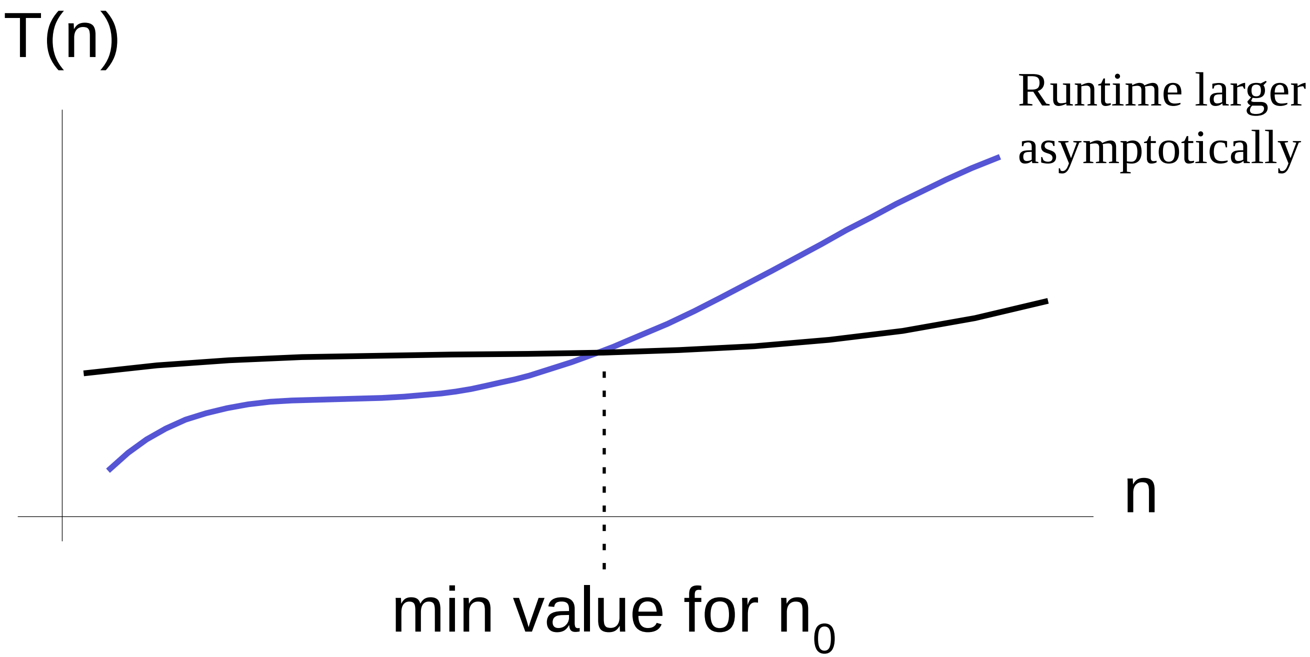
e.g.

$$2\boldsymbol{n}^{\boldsymbol{2}}+5n+3 = \Theta(n^2)$$

$$3\boldsymbol{n}^{\boldsymbol{3}}+90n^2-2n+5 = \Theta(n^3)$$

- *Formal explanations in the next lecture.*

- As n gets large, a $\Theta(n^2)$ algorithm runs faster than a $\Theta(n^3)$ algorithm



Insertion Sort – Runtime Analysis

Cost	Insertion-Sort (A)
c_1	1. **for** j ← 2 **to** n **do**
c_2	2. key ← A[j];
c_3	3. i ← j - 1;
c_4	4. **while** i > 0 **and** A[i] > key **do**
c_5	5. A[i+1] ← A[i];
c_6	6. i ← i - 1;
	endwhile
c_7	7. A[i+1] ← key;
	endfor

t_j: The number of times while loop test is executed for j

How many times is each line executed?

# times	Insertion-Sort (A)
n	1. **for** j ← 2 **to** n **do**
n-1	2. key ← A[j];
n-1	3. i ← j - 1;
k_4	4. **while** i > 0 **and** A[i] > key **do**
k_5	5. A[i+1] ← A[i];
k_6	6. i ← i - 1;
	endwhile
n-1	7. A[i+1] ← key;
	endfor

$$k_4 = \sum_{j=2}^{n} t_j$$

$$k_5 = \sum_{j=2}^{n} (t_j - 1)$$

$$k_6 = \sum_{j=2}^{n} (t_j - 1)$$

Insertion Sort – Runtime Analysis

- Sum up costs:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j +$$
$$c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

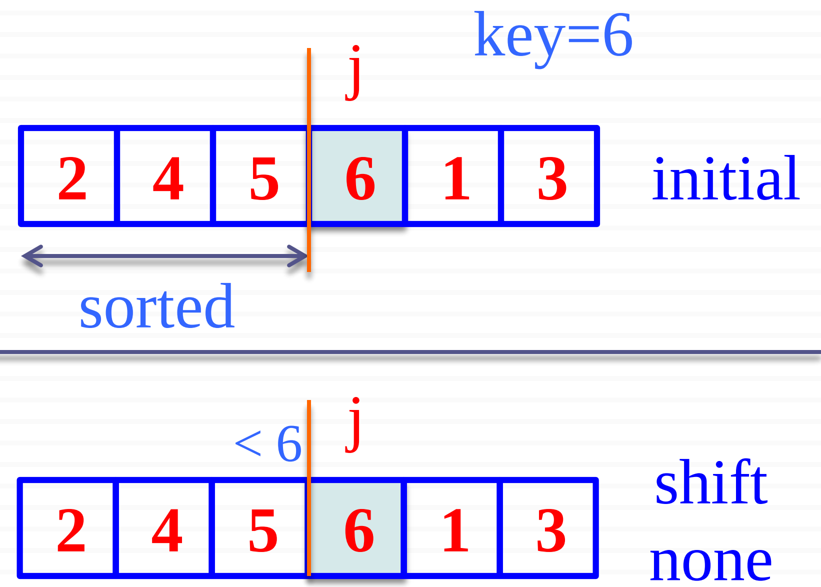
- What is the best case runtime?

- What is the worst case runtime?

Question: If $A[1...j]$ is already sorted, $t_j = ?$

Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
   do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```



$t_j = 1$

Insertion Sort – Best Case Runtime

- Original function:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j - 1) + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7(n-1)$$

- Best-case: Input array is already sorted

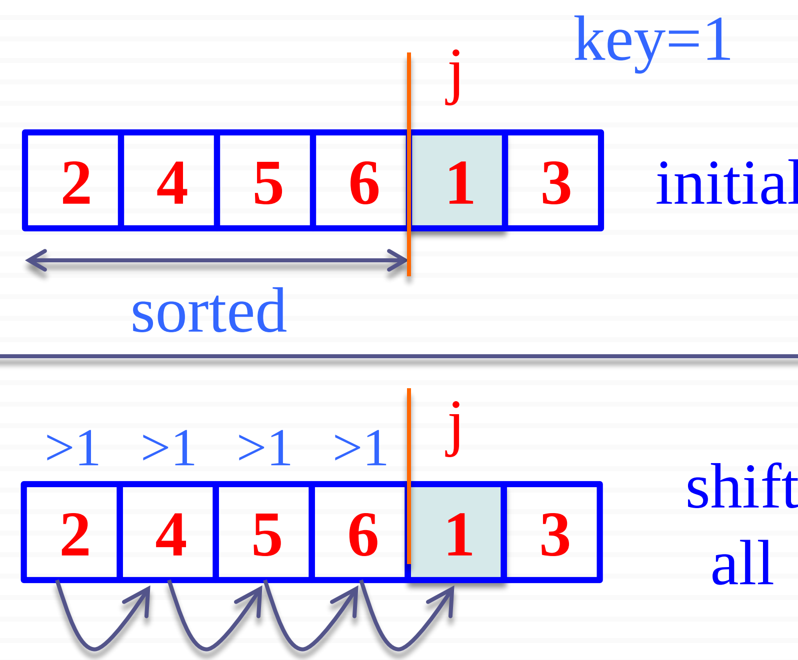
$t_j = 1$ for all j

$$T(n) = (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

Q: If A[j] is smaller than every entry in A[1..j-1], t_j = ?

Insertion-Sort (A)

```
for j ← 2 to n do
    key ← A[j];
    i ← j - 1;
    while i > 0 and A[i] > key
     do
        A[i+1] ← A[i];
        i ← i - 1;
    endwhile
    A[i+1] ← key;
endfor
```



$$t_j = j$$

Insertion Sort – Worst Case Runtime

- Worst case: The input array is reverse sorted

  $t_j = j$ for all j

- After derivation, worst case runtime:

$$T(n) = \tfrac{1}{2}(c_4 + c_5 + c_6)n^2 +$$
$$(c_1 + c_2 + c_3 + \tfrac{1}{2}(c_4 - c_5 - c_6) + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

Insertion Sort – Asymptotic Runtime Analysis

<u>Insertion-Sort</u> (A)

```
for j ← 2 to n do
    key ← A[j];                      Θ(1)
    i ← j - 1;
    while i > 0 and A[i] > key
    do
        A[i+1] ← A[i];               Θ(1)
        i ← i - 1;
    endwhile
    A[i+1] ← key;                    Θ(1)
endfor
```

Asymptotic Runtime Analysis of Insertion-Sort

- Worst-case (input reverse sorted)
  - *Inner loop is $\Theta(j)$*

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta\left(\sum_{j=2}^{n} j\right) = \Theta(n^2)$$

- Average case (all permutations equally likely)
  - *Inner loop is $\Theta(j/2)$*

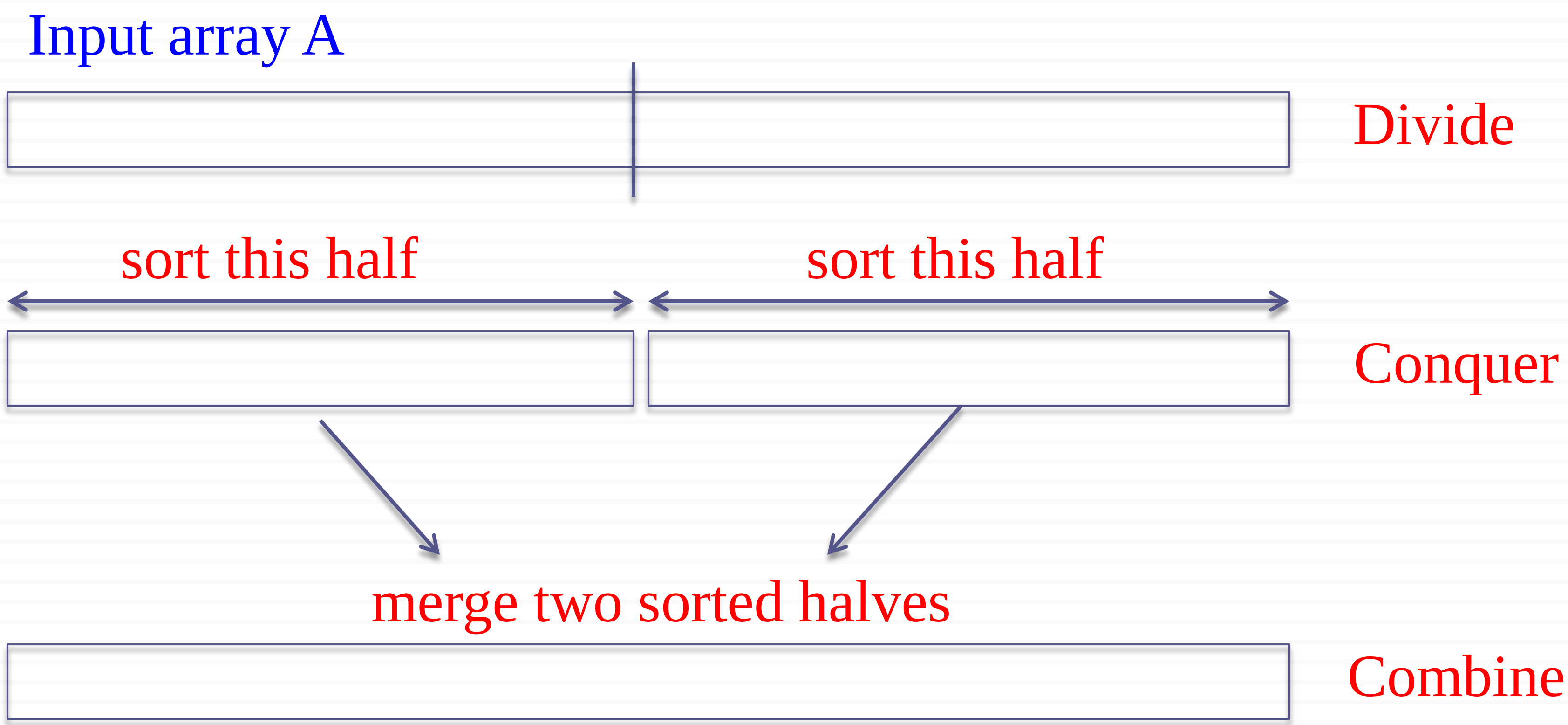
$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

    - Often, average case not much better than worst case

- Is this a fast sorting algorithm?
  - Yes, for small n. No, for large n.

Merge Sort

Merge Sort: Basic Idea



Merge-Sort (A, p, r)

```
if p = r then return;
else
    q ← ⌊(p+r)/2⌋;                      (Divide)

    Merge-Sort (A, p, q);               (Conquer)
    Merge-Sort (A, q+1, r);             (Conquer)
    Merge (A, p, q, r);                 (Combine)
endif
```

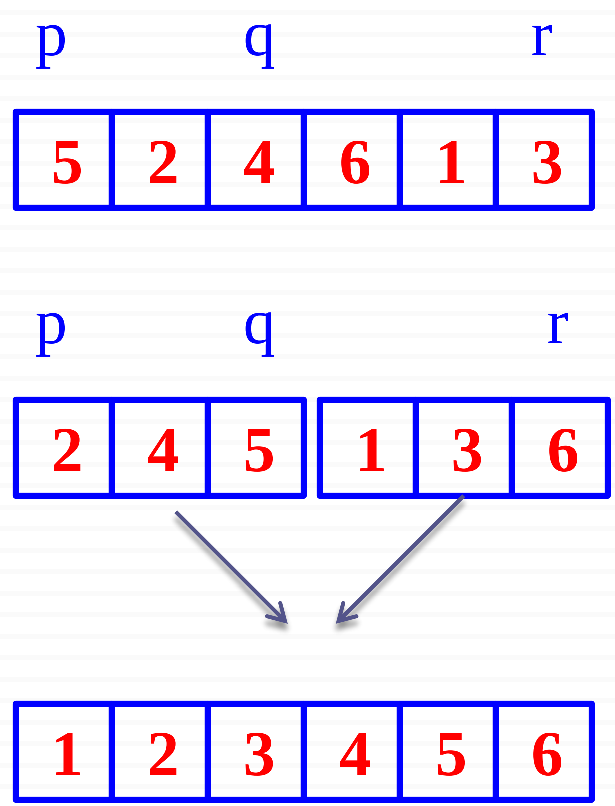
- Call Merge-Sort(A,1,n) to sort A[1..n]
- Recursion bottoms out when subsequences have length 1

Merge Sort: Example

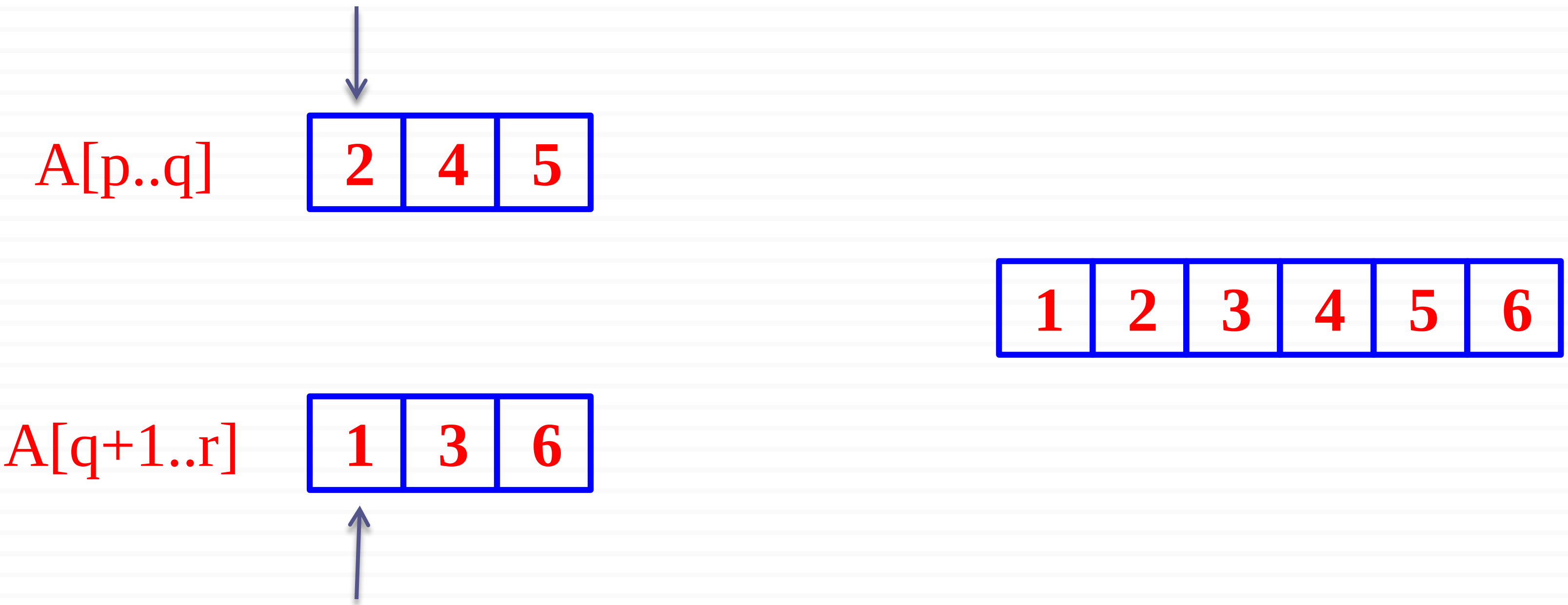
Merge-Sort (A, p, r)
    if p = r **then**
        return
    else
        q ← $\lfloor (p+r)/2 \rfloor$

        Merge-Sort (A, p, q)
        Merge-Sort (A, q+1, r)

        Merge(A, p, q, r)
    endif



How to merge 2 sorted subarrays?



- *HW: Study the pseudo-code in the textbook (Sec. 2.3.1)*
- What is the complexity of this step? $\Theta(n)$

Merge Sort: Correctness

Merge-Sort (A, p, r)
if p = r **then**
return
else
q ← ⌊ (p+r)/2⌋

Merge-Sort (A, p, q)
Merge-Sort (A, q+1, r)

Merge(A, p, q, r)
endif

Base case: p = r
→ Trivially correct

Inductive hypothesis: MERGE-SORT is correct for any subarray that is a ***strict*** (smaller) ***subset*** of A[p, q].

General Case: MERGE-SORT is correct for A[p, q].
→ From inductive hypothesis and correctness of *Merge*.

Merge Sort: Complexity

Merge-Sort (A, p, r)	→ $T(n)$
if p = r **then** **return**	→ $\Theta(1)$
else	
$q \leftarrow \lfloor (p+r)/2 \rfloor$	→ $\Theta(1)$
Merge-Sort (A, p, q)	→ $T(n/2)$
Merge-Sort (A, q+1, r)	→ $T(n/2)$
Merge(A, p, q, r)	→ $\Theta(n)$
endif	

Merge Sort – Recurrence

- Describe a function recursively in terms of itself
- To analyze the performance of recursive algorithms

- For merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

How to solve for T(n)?

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

- Generally, we will assume $T(n) = \Theta(1)$ for sufficiently small n
- The recurrence above can be rewritten as:

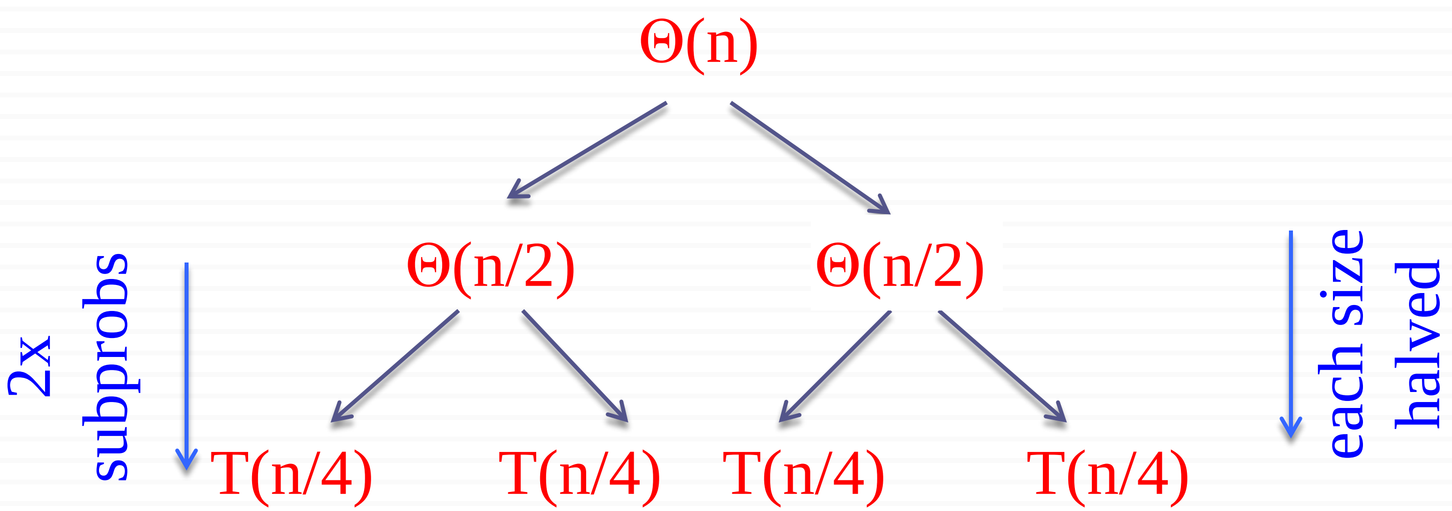
$$T(n) = 2\,T(n/2) + \Theta(n)$$

- How to solve this recurrence?

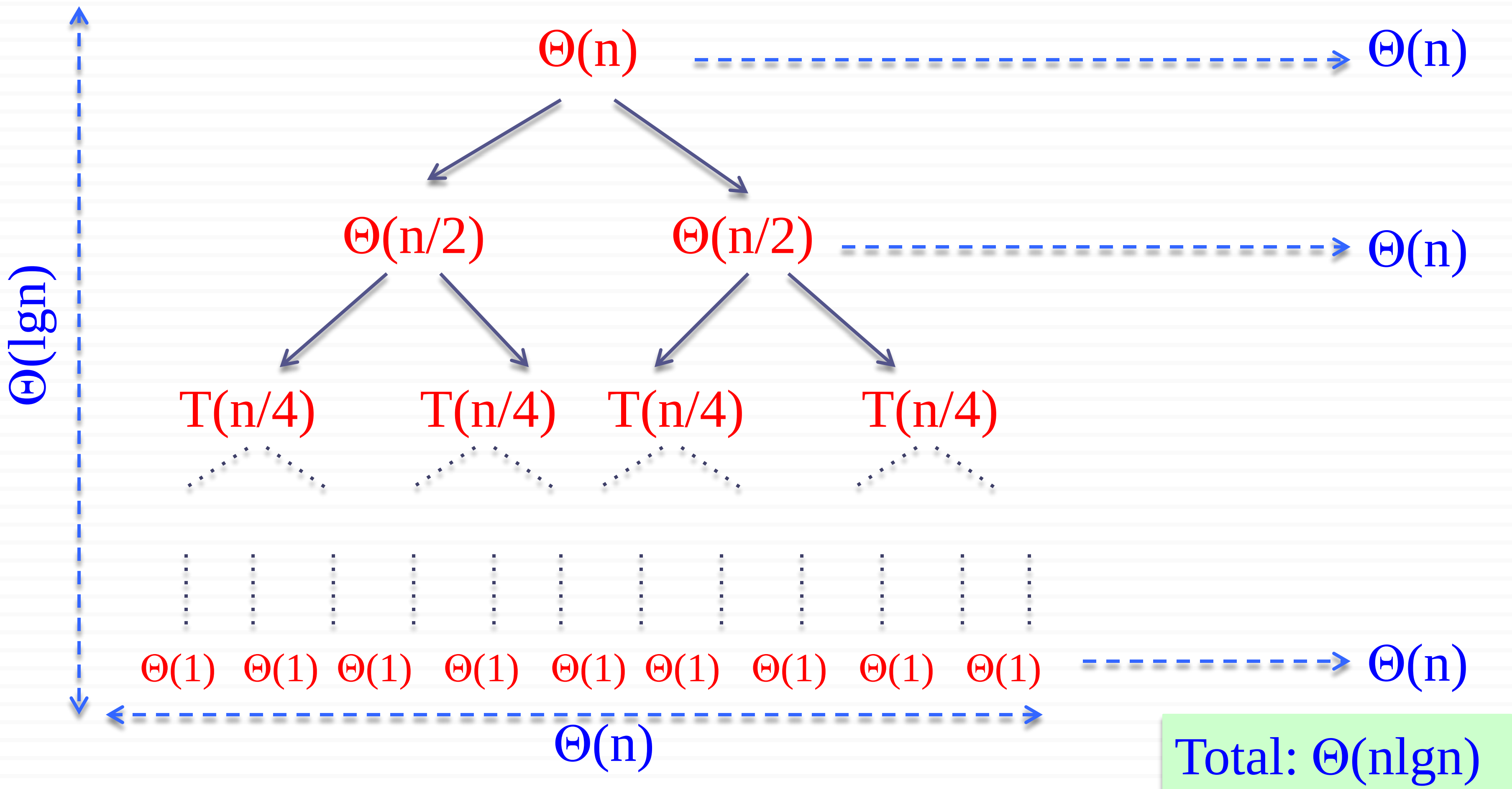
Solve Recurrence: $T(n) = 2T\,(n/2) + \Theta(n)$

$$
\begin{tikzcd}[column sep=small]
 & \Theta(n) \arrow[dl] \arrow[dr] & \\
T(n/2) & & T(n/2)
\end{tikzcd}
$$

Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Solve Recurrence: $T(n) = 2T(n/2) + \Theta(n)$



Merge Sort Complexity

- Recurrence:

  $T(n) = 2T(n/2) + \Theta(n)$

- Solution to recurrence:

  $T(n) = \Theta(n\lg n)$

Conclusions: Insertion Sort vs. Merge Sort

- $\Theta(nlgn)$ grows more slowly than $\Theta(n^2)$

- Therefore Merge-Sort beats Insertion-Sort in the worst case

- In practice, Merge-Sort beats Insertion-Sort for $n>30$ or so.